

Telescoping:

Telescoping is a method of cancelling terms when calculating infinite sums or products.

Example 1:

To find $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$, we observe that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$, so

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{k(k+1)} &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k(k+1)} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right) = \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \dots + \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} \right) = \\ &= \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1} \right) = 1\end{aligned}$$

Practice: Prove that $\sum_{k=1}^{\infty} \frac{1}{n(n+2)} = \frac{3}{4}$

Example 2: To find $\sum_{k=1}^n \ln \left(\frac{k+1}{k} \right)$ we use logarithmic properties and telescoping:

$$\begin{aligned}\sum_{k=1}^n \ln \left(\frac{k+1}{k} \right) &= \ln \left(\prod_{k=1}^n \frac{k+1}{k} \right) = \ln \left(\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \dots \cdot \frac{n}{n-1} \cdot \frac{n+1}{n} \right) = \ln(n+1), \text{ so the series} \\ \sum_{k=1}^{\infty} \ln \left(\frac{k+1}{k} \right) &\text{ diverges.}\end{aligned}$$

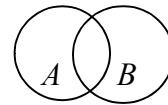
Inclusion-Exclusion Principle:

Let $\{A_i\}_{i=1}^n$ be a system of n sets $A_1, \dots, A_n$, then

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i_1 \leq i_2 \leq n} |A_{i_1} \cap A_{i_2}| + \sum_{1 \leq i_1 \leq i_2 \leq i_3 \leq n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \dots + (-1)^{n-1} \left| \bigcap_{i=1}^n A_i \right|,$$

where $|A|$ represents cardinality of set A .

If $n = 3$, then we have $|A \cup B| = |A| + |B| - |A \cap B|$



Example 3: How many positive integers not exceeding 1000 are divisible by 7 or 11?

Let A be the set of such integers divisible by 7 and B - divisible by 11, then $A \cup B$ are the once divisible by 7 or 11. We have $|A| = \left\lfloor \frac{1000}{7} \right\rfloor = 142$, $|B| = \left\lfloor \frac{1000}{11} \right\rfloor = 90$, and $|A \cap B| = \left\lfloor \frac{1000}{77} \right\rfloor = 12$.

Therefore $|A \cup B| = |A| + |B| - |A \cap B| = 142 + 90 - 12 = 220$.