

PUTNAM TRAINING PROBLEMS, 2005

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REMARK. This is a list of problems discussed during the training sessions of the NU Putnam team and arranged by subjects. —Miguel A. Lerma

EXERCISES

1. Induction.

- 1.1. Prove that $n! > 2^n$ for all $n \geq 4$.
- 1.2. Prove that for any integer $n \geq 1$, $2^{2^n} - 1$ is divisible by 3.
- 1.3. Let a and b two distinct integers, and n any positive integer. Prove that $a^n - b^n$ is divisible by $a - b$.
- 1.4. The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \dots$ is defined as a sequence whose two first terms are $F_0 = 0$, $F_1 = 1$ and each subsequent term is the sum of the two previous ones: $F_n = F_{n-1} + F_{n-2}$ (for $n \geq 2$). Prove that $F_n < 2^n$ for every $n \geq 0$.
- 1.5. Let r be a number such that $r + 1/r$ is an integer. Prove that for every positive integer n , $r^n + 1/r^n$ is an integer.
- 1.6. Find the number $R(n)$ of regions in which the plane can be divided by n straight lines.
- 1.7. We divide the plane into regions using straight lines. Prove that those regions can be colored with two colors so that no two regions that share a boundary have the same color.
- 1.8. A great circle is a circle drawn on a sphere that is an “equator”, i.e., its center is also the center of the sphere. There are n great circles on a sphere, no three of which meet at any point. They divide the sphere into how many regions?
- 1.9. We need to put n cents of stamps on an envelop, but we have only (an unlimited supply of) 5¢ and 12¢ stamps. Prove that we can perform the task if $n \geq 44$.
- 1.10. A chessboard is a 8×8 grid (64 squares arranged in 8 rows and 8 columns), but here we will call “chessboard” any $m \times m$ square grid. We call *defective* a chessboard if one of its squares is missing. Prove that any $2^n \times 2^n$ ($n \geq 1$) defective chessboard can be

tiled (completely covered without overlapping) with L-shaped trominos occupying exactly 3 squares, like this .

- 1.11.** This is a modified version of the game of Nim (in the following we assume that there is an unlimited supply of tokens.) Two players arrange several piles of tokens in a row. By turns each of them takes one token from one of the piles and adds at will as many tokens as he or she wishes to piles placed to the left of the pile from which the token was taken. Assuming that the game ever finishes, the player that takes the last token wins. Prove that, no matter how they play, the game will eventually end after finitely many steps.
- 1.12.** Call an integer square-full if each of its prime factors occurs to a second power (at least). Prove that there are infinitely many pairs of consecutive square-fulls.
- 1.13.** Prove that for every $n \geq 2$, the expansion of $(1 + x + x^2)^n$ contains at least one even coefficient.

2. Inequalities.

2.1. If $a, b, c > 0$, prove that $(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq 9a^2b^2c^2$.

2.2. Prove that $n! < \left(\frac{n+1}{2}\right)^n$, for $n = 2, 3, 4, \dots$,

2.3. If $0 < p, 0 < q$, and $p + q < 1$, prove that $(px + qy)^2 \leq px^2 + qy^2$.

2.4. If $a, b, c \geq 0$, prove that $\sqrt{3(a+b+c)} \geq \sqrt{a} + \sqrt{b} + \sqrt{c}$.

2.5. Let $x, y, z > 0$ with $xyz = 1$. Prove that $x + y + z \leq x^2 + y^2 + z^2$.

2.6. Show that

$$\sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2} + \dots + \sqrt{a_n^2 + b_n^2} \geq \sqrt{(a_1 + a_2 + \dots + a_n)^2 + (b_1 + b_2 + \dots + b_n)^2}$$

2.7. Find the minimum value of the function $f(x_1, x_2, \dots, x_n) = x_1 + x_2 + \dots + x_n$, where $x_1, x_2, \dots, x_n$ are positive real numbers such that $x_1 x_2 \dots x_n = 1$.

2.8. Let $x, y, z \geq 0$ with $xyz = 1$. Find the minimum of

$$S = \frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}.$$

2.9. If $x, y, z > 0$, and $x + y + z = 1$, find the minimum value of

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

- 2.10.** Prove that in a triangle with sides a, b, c and opposite angles A, B, C (in radians) the following relation holds:

$$\frac{aA + bB + cC}{a + b + c} \geq \frac{\pi}{3}.$$

- 2.11.** (Putnam, 2003) Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ nonnegative real numbers. Show that

$$(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \leq ((a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n))^{1/n}$$

- 2.12.** The notation $n!^{(k)}$ means take factorial of n k times. For example, $n!^{(3)}$ means $((n!)!)!$. What is bigger, $1999!^{(2000)}$ or $2000!^{(1999)}$?

- 2.13.** Which is larger, 1999^{1999} or 2000^{1998} ?

- 2.14.** Prove that there are no positive integers a, b such that $b^2 + b + 1 = a^2$.

- 2.15.** (Inspired in Putnam 1968, B6) Prove that a polynomial with only real roots and all coefficients equal to ± 1 has degree at most 3.

- 2.16.** (Putnam 1984) Find the minimum value of

$$(u - v)^2 + \left(\sqrt{2 - u^2} - \frac{9}{v} \right)^2$$

for $0 < u < \sqrt{2}$ and $v > 0$.

- 2.17.** Show that $\frac{1}{\sqrt{4n}} \leq \left(\frac{1}{2}\right) \left(\frac{3}{4}\right) \cdots \left(\frac{2n-1}{2n}\right) < \frac{1}{\sqrt{2n}}$.

- 2.18.** (Putnam, 2004) Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \frac{n!}{n^n}.$$

- 2.19.** Let $a_1, a_2, \dots, a_n$ be a sequence of positive numbers, and let $b_1, b_2, \dots, b_n$ be any permutation of the first sequence. Show that

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \geq n.$$

- 2.20.** Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ increasing sequences of real numbers, and let $x_1, x_2, \dots, x_n$ be any permutation of $b_1, b_2, \dots, b_n$. Show that

$$\sum_{i=1}^n a_i b_i \geq \sum_{i=1}^n a_i x_i.$$

3. Number Theory.

- 3.1.** Show that the sum of two consecutive primes is never twice a prime.

- 3.2.** Prove that there are infinitely many prime numbers of the form $4n + 3$.
- 3.3.** Prove that the fraction $(n^3 + 2n)/(n^4 + 3n^2 + 1)$ is in lowest terms for every possible integer n .
- 3.4.** Let $p(x)$ be a non-constant polynomial such that $p(n)$ is an integer for every positive integer n . Prove that $p(n)$ is composite for infinitely many positive integers n . (This proves that there is no polynomial yielding only prime numbers.)
- 3.5.** Prove that two consecutive Fibonacci numbers are always relatively prime.
- 3.6.** Show that if $a^2 + b^2 = c^2$, then $3|ab$.
- 3.7.** Show that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ can never be an integer for $n \geq 2$.
- 3.8.** Let $f(n)$ denote the sum of the digits of n . Let $N = 4444^{4444}$. Find $f(f(f(N)))$.
- 3.9.** Show that there exist 1999 consecutive numbers, each of which is divisible by the cube of an integer.
- 3.10.** Find all triples of positive integers (a, b, c) such that
- $$\left(1 + \frac{1}{a}\right) \left(1 + \frac{1}{b}\right) \left(1 + \frac{1}{c}\right) = 2.$$
- 3.11.** Find all positive integers solutions to $abc - 2 = a + b + c$.
- 3.12.** (USAMO, 1979) Find all non-negative integral solutions $(n_1, n_2, \dots, n_{14})$ to
- $$n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1599.$$
- 3.13.** The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \dots$ is defined by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove that for some $k > 0$, F_k is a multiple of $10^{10^{10}}$.

4. Polynomials.

- 4.1.** Find a polynomial with integral coefficients whose zeros include $\sqrt{2} + \sqrt{5}$.
- 4.2.** Prove that the sum
- $$\sqrt{1001^2 + 1} + \sqrt{1002^2 + 1} + \cdots + \sqrt{2000^2 + 1}$$
- is irrational.
- 4.3.** (USAMO 1975) If $P(x)$ denotes a polynomial of degree n such that $P(k) = k/(k+1)$ for $k = 0, 1, 2, \dots, n$, determine $P(n+1)$.
- 4.4.** (USAMO 1984) The product of two of the four zeros of the quartic equation
- $$x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$$

is -32 . Find k .

4.5. Let $p(x)$ be a polynomial with integer coefficients satisfying that $p(0)$ and $p(1)$ are odd. Show that p has no integer zeros.

4.6. (USAMO 1976) If $P(x)$, $Q(x)$, $R(x)$, $S(x)$ are polynomials such that

$$P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + x^2 + x + 1)S(x)$$

prove that $x - 1$ is a factor of $P(x)$.

4.7. Let a, b, c distinct integers. Can the polynomial $(x - a)(x - b)(x - c) - 1$ be factored into the product of two polynomials with integer coefficients?

4.8. Let $p_1, p_2, \dots, p_n$ distinct integers and let $f(x)$ be the polynomial of degree n given by

$$f(x) = (x - p_1)(x - p_2) \cdots (x - p_n).$$

Prove that the polynomial

$$g(x) = (f(x))^2 + 1$$

cannot be expressed as the product of two non-constant polynomials with integral coefficients.

4.9. Find the remainder when you divide $x^{81} + x^{49} + x^{25} + x^9 + x$ by $x^3 - x$.

5. Complex Numbers.

5.1. Let m and n two integers such that each can be expressed as the sum of two perfect squares. Prove that mn has this property as well. For instance $17 = 4^2 + 1^2$, $13 = 2^2 + 3^2$, and $17 \cdot 13 = 221 = 14^2 + 5^2$.

5.2. Prove that $\sum_{k=0}^n \sin k = \frac{\sin \frac{n}{2} \sin \frac{n+1}{2}}{\sin \frac{1}{2}}$.

5.3. Show that if $x + 1/x = 2 \cos a$, then for any integer n , $x^n + 1/x^n = 2 \cos na$.

5.4. Factor $p(z) = z^5 + z + 1$.

5.5. Find a close-form expression for $\prod_{k=1}^{n-1} \sin \frac{k\pi}{n}$.

5.6. Consider a regular n -gon which is inscribed in a circle with radius 1. What is the product of the lengths of all $n(n-1)/2$ diagonals of the polygon (this includes the sides of the n -gon).

- 5.7.** (Putnam 1991, B2) Suppose f and g are nonconstant, differentiable, real-valued functions on $\mathbb{R}$. Furthermore, suppose that for each pair of real numbers x and y

$$f(x+y) = f(x)f(y) - g(x)g(y)$$

$$g(x+y) = f(x)g(y) + g(x)f(y)$$

If $f'(0) = 0$ prove that $f(x)^2 + g(x)^2 = 1$ for all x .

- 5.8.** Given a circle of n lights, exactly one of which is initially on, it is permitted to change the state of a bulb provided that one also changes the state of every d th bulb after it (where d is a divisor of n strictly less than n), provided that all n/d bulbs were originally in the same state as one another. For what values of n is it possible to turn all the bulbs on by making a sequence of moves of this kind?

6. Generating Functions.

- 6.1.** Prove that for any positive integer n

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1},$$

where $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ (binomial coefficient).

- 6.2.** Prove that for any positive integer n

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

- 6.3.** Prove that for any positive integers $k \leq m, n$,

$$\sum_{j=0}^k \binom{n}{j} \binom{m}{k-j} = \binom{m+n}{k}.$$

- 6.4.** Let F_n be the Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \dots$, defined recursively $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove that

$$\sum_{n=1}^{\infty} \frac{F_n}{2^n} = 2.$$

- 6.5.** How many distinct ordered pairs (a, b) of nonnegative integers satisfy

$$2a + 5b = 100?$$

- 6.6.** How many different sequences are there that satisfy all the following conditions:

- (a) The items of the sequences are the digits 0–9.
- (b) The length of the sequences is 6 (e.g. 061030)
- (c) Repetitions are allowed.
- (d) The sum of the items is exactly 10 (e.g. 111322).

- 6.7.** (Leningrad Mathematical Olympiad 1991) A finite sequence $a_1, a_2, \dots, a_n$ is called p -balanced if any sum of the form $a_k + a_{k+p} + a_{k+2p} + \dots$ is the same for any $k = 1, 2, 3, \dots, p$. For instance the sequence $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4, a_5 = 3, a_6 = 2$ is 3-balanced because $a_1 + a_4 = 1 + 4 = 5, a_2 + a_5 = 2 + 3 = 5, a_3 + a_6 = 3 + 2 = 5$. Prove that if a sequence with 50 members is p -balanced for $p = 3, 5, 7, 11, 13, 17$, then all its members are equal zero.

7. Recurrences.

- 7.1.** Find the number of subsets of $\{1, 2, \dots, n\}$ that contain no two consecutive elements of $\{1, 2, \dots, n\}$.
- 7.2.** Determine the maximum number of regions in the plane that are determined by n “vee”s. A “vee” is two rays which meet at a point. The angle between them is any positive number.
- 7.3.** Define a *domino* to be a 1×2 rectangle. In how many ways can an $n \times 2$ rectangle be tiled by dominoes?
- 7.4.** (Putnam 1996) Define a *selfish* set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \dots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets are selfish.
- 7.5.** Let $a_1, a_2, \dots, a_n$ be an ordered sequence of n distinct objects. A *derangement* of this sequence is a permutation that leaves no object in its original place. For example, if the original sequence is 1, 2, 3, 4, then 2, 4, 3, 1 is not a derangement, but 2, 1, 4, 3 is. Let D_n denote the number of derangements of an n -element sequence. Show that

$$D_n = (n - 1)(D_{n-1} + D_{n-2}).$$

8. Calculus.

- 8.1.** Believe it or not the following function is constant in an interval $[a, b]$. Find that interval and the constant value of the function.

$$f(x) = \sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}}.$$

- 8.2.** Find the value of the following infinitely nested radical

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}}.$$

8.3. (Putnam 1995) Evaluate

$$\sqrt[8]{2207 - \frac{1}{2207 - \frac{1}{2207 - \dots}}}$$

Express your answer in the form $(a + b\sqrt{c})/d$, where a, b, c, d , are integers.

8.4. (Putnam 1992) Let f be an infinitely differentiable real-valued function defined on the real numbers. If

$$f\left(\frac{1}{n}\right) = \frac{n^2}{n^2 + 1}, \quad n = 1, 2, 3, \dots$$

compute the values of the derivatives $f^{(k)}(0)$, $k = 1, 2, 3, \dots$.

8.5. Compute $\lim_{n \rightarrow \infty} \left\{ \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n-1} \right\}$.

8.6. Compute $\lim_{n \rightarrow \infty} \left\{ \prod_{k=1}^n \left(1 + \frac{k}{n} \right) \right\}^{1/n}$.

8.7. (Putnam 1997) Evaluate

$$\int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} \right) \left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} \right) dx.$$

8.8. (Putnam 1990) Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$ ($n, m = 0, 1, 2, \dots$)? (In other words, is it possible to find integers n and m such that $\sqrt[3]{n} - \sqrt[3]{m}$ is as close as we wish to $\sqrt{2}$?)

8.9. (Leningrad Mathematical Olympiad, 1988) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous, with $f(x) \cdot f(f(x)) = 1$ for all $x \in \mathbb{R}$. If $f(1000) = 999$, find $f(500)$.

8.10. Let $f : [0, 1] \rightarrow \mathbb{R}$ continuous, and suppose that $f(0) = f(1)$. Show that there is a value $x \in [0, 1998/1999]$ satisfying $f(x) = f(x + 1/1999)$.

9. Pigeonhole Principle.

9.1. Prove that any $(n+1)$ -element subset of $\{1, 2, \dots, 2n\}$ contains two integers that are relatively prime.

9.2. Prove that if we select $n+1$ numbers from the set $S = \{1, 2, 3, \dots, 2n\}$, among the numbers selected there are two such that one is a multiple of the other one.

9.3. (Putnam 1978) Let A be any set of 20 distinct integers chosen from the arithmetic progression $\{1, 4, 7, \dots, 100\}$. Prove that there must be two distinct integers in A whose sum is 104.

- 9.4.** Let A be the set of all 8-digit numbers in base 3 (so they are written with the digits 0,1,2 only), including those with leading zeroes such as 00120010. Prove that given 4 elements from A , two of them must coincide in at least 2 places.
- 9.5.** During a month with 30 days a baseball team plays at least a game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

10. Telescoping.

- 10.1.** (Putnam 1984) Express

$$\sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$$

as a rational number.

- 10.2.** (Putnam 1977) Evaluate the infinite product

$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}.$$

11. Symmetries.

- 11.1.** A spherical, 3-dimensional planet has center at $(0, 0, 0)$ and radius 20. At any point of the surface of this planet, the temperature is $T(x, y, z) = (x + y)^2 + (y - z)^2$ degrees. What is the average temperature of the surface of this planet?
- 11.2.** (Putnam 1980) Evaluate $\int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}}.$

12. Inclusion-Exclusion.

- 12.1.** How many positive integers not exceeding 1000 are divisible by 7 or 11?
- 12.2.** Imagine that you are going to give n kids ice-cream cones, one cone per kid, and there are k different flavors available. Assuming that no flavor gets mixed, find the number of ways we can give out the cones using all k flavors.
- 12.3.** Let $a_1, a_2, \dots, a_n$ an ordered sequence of n distinct objects. A *derangement* of this sequence is a permutation that leaves no object in its original place. For example, if the original sequence is $\{1, 2, 3, 4\}$, then $\{2, 4, 3, 1\}$ is not a derangement, but $\{2, 1, 4, 3\}$ is. Let D_n denote the number of derangements of an n -element sequence. Show that

$$D_n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \dots + (-1)^n \frac{1}{n!} \right).$$

13. Miscellany.

- 13.1.** (Putnam 1986) What is the units (i.e., rightmost) digit of $\left\lfloor \frac{10^{20000}}{10^{100} + 3} \right\rfloor$?
- 13.2.** (IMO 1975) Prove that there are infinitely many points on the unit circle $x^2 + y^2 = 1$ such that the distance between any two of them is a rational number.
- 13.3.** (Putnam 1988) Prove that if we paint every point of the plane in one of three colors, there will be two points one inch apart with the same color. Is this result necessarily true if we replace "three" by "nine"?
- 13.4.** Imagine an infinite chessboard that contains a positive integer in each square. If the value of each square is equal to the average of its four neighbors to the north, south, west and east, prove that the values in all the squares are equal.
- 13.5.** (Putnam 1990) Consider a paper punch that can be centered at any point of the plane and that, when operated, removes precisely those points whose distance from the center is irrational. How many punches are needed to remove every point?
- 13.6.** (Putnam 1984) Let n be a positive integer, and define
- $$f(n) = 1! + 2! + \cdots + n!.$$
- Find polynomials $P(x)$ and $Q(x)$ such that
- $$f(n+2) = P(n)f(n+1) + Q(n)f(n)$$
- for all $n \geq 1$.
- 13.7.** (Putnam 1974) Call a set of positive integers "conspiratorial" if no three of them are pairwise relatively prime. What is the largest number of elements in any conspiratorial subset of integers 1 through 16?
- 13.8.** (Putnam 1984) Prove or disprove the following statement: If F is a finite set with two or more elements, then there exists a binary operation $*$ on F such that for all x, y, z in F ,
- (i) $x * z = y * z$ implies $x = y$ (right cancellation holds), and
 - (ii) $x * (y * z) \neq (x * y) * z$ (no case of associativity holds).

HINTS

- 1.1. —
- 1.2. For the induction step, rewrite $2^{2(n+1)} - 1$ as a sum of two terms that are divisible by 3.
- 1.3. For the inductive step assume that step $a^n - b^n$ is divisible by $a - b$ and rewrite $a^{n+1} - b^{n+1}$ as a sum of two terms, one of them involving $a^n - b^n$ and the other one being a multiple of $a - b$.
- 1.4. Strong induction.
- 1.5. Rewrite $r^{n+1} + 1/r^{n+1}$ in terms of $r^k + 1/r^k$ with $k \leq n$.
- 1.6. How many regions can be intersected by the $(n + 1)$ th line?
- 1.7. Color a plane divided with n of lines in the desired way, and think how to recolor it after introducing the $(n + 1)$ th line.
- 1.8. How many regions can be intersected the by $(n + 1)$ th circle?
- 1.9. We have $1 = 5 \cdot (-7) + 12 \cdot 6 = 5 \cdot 5 + 12 \cdot (-2)$. Also, prove that if $n = 5x + 12y \geq 44$, then either $x \geq 7$ or $y \geq 2$.
- 1.10. For the inductive step, consider a $2^{n+1} \times 2^{n+1}$ defective chessboard and divide it into four $2^n \times 2^n$ chessboards. One of them is defective. Can the other three be made defective by placing strategically an L?
- 1.11. Use induction on the number of piles.
- 1.12. The numbers 8 and 9 form one such pair. Given a pair $(n, n + 1)$ of consecutive square-fulls, find some way to build another pair of consecutive square-fulls.
- 1.13. Look at oddness/evenness of the four lowest degree terms of the expansion.
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- 2.1. One way to solve this problem is by using the AGM inequality on each factor of the left hand side.
- 2.2. Apply the AGM inequality to the set of numbers $1, 2, \dots, n$.
- 2.3. Power means inequality with weights $\frac{p}{p+q}$ and $\frac{q}{p+q}$.
- 2.4. Power means inequality.
- 2.5. —

- 2.6.** This problem can be solved by using Minkowski's inequality, but another way to look at it is by an appropriate geometrical interpretation of the terms (as distances between points of the plane.)
- 2.7.** Many minimization or maximization problems are inequalities in disguise. The solution usually consists of "guessing" the maximum or minimum value of the function, and then proving that it is in fact maximum or minimum. In this case, given the symmetry of the function a good guess is $f(1, 1, \dots, 1) = n$, so try to prove $f(x_1, x_2, \dots, x_n) \geq n$. Use the AGM inequality on $x_1, \dots, x_n$.
- 2.8.** Apply the Cauchy-Schwarz inequality to the vectors $(\frac{x}{\sqrt{y+z}}, \frac{y}{\sqrt{z+x}}, \frac{z}{\sqrt{x+y}})$ and (u, v, w) , and choose appropriate values for u, v, w .
- 2.9.** Arithmetic-Harmonic Mean inequality.
- 2.10.** Assume $a \leq b \leq c$, $A \leq B \leq C$, and use Chebyshev's Inequality.
- 2.11.** Divide by the right hand side and use the AGM inequality on both terms of the left.
- 2.12.** Note that $n!$ is increasing ($n < m \implies n! < m!$)
- 2.13.** Look at the function $f(x) = (1999 - x) \ln(1999 + x)$.
- 2.14.** The numbers b^2 and $(b + 1)^2$ are consecutive squares.
- 2.15.** Use the AGM inequality on the squares of the roots of the polynomial.
- 2.16.** Think geometrically. Interpret the given expression as the square of the distance between two points in the plane. The problem becomes that of finding the minimum distance between two curves.
- 2.17.** Consider the expressions $P = (\frac{1}{2}) (\frac{3}{4}) \cdots (\frac{2n-1}{2n})$ and $Q = (\frac{2}{3}) (\frac{4}{5}) \cdots (\frac{2n-2}{2n-1})$. Note that $\frac{k-1}{k} < \frac{k}{k+1}$, for $k = 1, 2, \dots$
- 2.18.** Look at the binomial expansion of $(m + n)^{m+n}$.
- 2.19.** AGM inequality.
- 2.20.** Try first the cases $n = 1$ and $n = 2$. Then use induction.
- 3.1.** Contradiction.
- 3.2.** Assume that there are finitely many primes of the form $4n + 3$, call P their product, and try to obtain a contradiction similar to the one in Euclid's proof of the infinitude of primes.
- 3.3.** Prove that $n^3 + 2n$ and $n^4 + 3n^2 + 1$ are relatively prime.

3.4. Prove that $p(k)$ divides $p(p(k) + k)$.

3.5. Induction.

3.6. Study the equation modulo 3.

3.7. Call the sum S and find the maximum power of 2 dividing each side of the equality

$$n!S = \sum_{k=1}^n \frac{n!}{k}.$$

3.8. $f(n) \equiv n \pmod{9}$.

3.9. Chinese Remainder Theorem.

3.10. The minimum of a, b, c cannot be very large.

3.11. Try changing variables $x = a + 1$, $y = b + 1$, $z = c + 1$.

3.12. Study the equation modulo 16.

3.13. Use the Pigeonhole Principle to prove that the sequence of pairs (F_n, F_{n+1}) is eventually periodic modulo $N = 10^{10^{10}}$.

4.1. Call $x = \sqrt{2} + \sqrt{5}$ and eliminate the radicals.

4.2. Prove that the sum is the root of a monic polynomial but not an integer.

4.3. Look at the polynomial $Q(x) = (x + 1)P(x) - x$.

4.4. Use the relationship between zeros and coefficients of a polynomial.

4.5. For each integer k study the parity of $p(k)$ depending on the parity of k .

4.6. We must prove that $P(1) = 0$. See what happens by replacing x with fifth roots of unity.

4.7. Assume $(x - a)(x - b)(x - c) - 1 = p(x)q(x)$, and look at the possible values of $p(x)$ and $q(x)$ for $x = a, b, c$.

4.8. Assume $g(x) = h(x)k(x)$, where $h(x)$ and $k(x)$ are non-constant polynomials with integral coefficients. Prove that they can be assumed to be positive for every x and $h(p_i) = k(p_i) = 1$, $i = 1, \dots, n$. Deduce that both are of degree n and determine their form. Get a contradiction by equating coefficients in $g(x)$ and $h(x)k(x)$.

4.9. The remainder will be a second degree polynomial. Plug the roots of $x^3 - x$.

5.1. If $m = a^2 + b^2$ and $n = c^2 + d^2$, then consider the product $z = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$.

- 5.2.** The left hand side of the equality is the imaginary part of $\sum_{k=0}^n e^{ik}$.
- 5.3.** Let $x = e^{ia}$.
- 5.4.** If $\omega = e^{2\pi i/3}$ then ω and ω^2 are two roots of $p(z)$.
- 5.5.** Write $\sin t = (e^{ti} - e^{-ti})/2i$.
- 5.6.** Assume the vertices of the n -gon placed on the complex plane at the n th roots of unity.
- 5.7.** Look at the function $h(x) = f(x) + ig(x)$.
- 5.8.** Assume the lights placed on the complex plane at the n th roots of unity $1, \zeta, \zeta^2, \dots, \zeta^{n-1}$, where $\zeta = e^{2\pi i/n}$.
- 6.1.** Expand and differentiate $(1+x)^n$.
- 6.2.** Expand both sides of $(1+x)^n(1+x)^n = (1+x)^{2n}$ and look at the coefficient of x^n .
- 6.3.** Expand both sides of $(1+x)^m(1+x)^n = (1+x)^{m+n}$ and look at the coefficient of x^j .
- 6.4.** Look at the generating function of the Fibonacci sequence.
- 6.5.** Find the generating function of the sequence u_n = number of nonnegative solutions of $2a + 5b = n$.
- 6.6.** The answer equals the coefficient of x^{10} in the expansion of $(1+x+x^2+\dots+x^9)^6$, but that coefficient is very hard to find directly. Try some simplification.
- 6.7.** Look at the polynomial $P(x) = a_1 + a_2x + a_3x^2 + \dots + a_{50}x^{49}$, and at its values at 3rd, 5th, ... roots of unity.
- 7.1.** The subsets of $\{1, 2, \dots, n\}$ that contain no two consecutive elements can be divided into two classes, the ones not containing n , and the ones containing n .
- 7.2.** The $(n+1)$ th “vee” divides the existing regions into how many further regions?
- 7.3.** The tilings of a $n \times 2$ rectangle by dominoes can be divided into two classes depending on whether we place the rightmost domino vertically or horizontally.
- 7.4.** The minimal selfish subsets of $\{1, 2, \dots, n\}$ can be divided into two classes depending on whether they contain n or not.

- 7.5. Assume that $b_1, b_2, \dots, b_n$ is a derangement of the sequence $a_1, a_2, \dots, a_n$. How many possible values can b_n have? Once we have fixed the value of b_n , divide the possible derangements into two appropriate classes.
- 8.1. $\sqrt{u^2} = |u|$.
- 8.2. Find the limit of the sequence $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$ ($n \geq 1$).
- 8.3. Call the limit L . Find some equation verified by L .
- 8.4. Justify that the desired derivatives must coincide with those of the function $g(x) = 1/(1 + x^2)$.
- 8.5. Compare the sum to some integral of the form $\int_a^b \frac{1}{x} dx$.
- 8.6. Take logarithms. Interpret the resulting expression as a Riemann sum.
- 8.7. Interpret the first series as a Maclaurin series. Interchange integration and summation with the second series (don't forget to justify why the interchange is "legitimate".)
- 8.8. In fact *any* real number r is the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$. We want $r \approx \sqrt[3]{n} - \sqrt[3]{m}$, i.e., $r + \sqrt[3]{m} \approx \sqrt[3]{n}$. Note that $\sqrt[3]{n+1} - \sqrt[3]{n} \rightarrow 0$ as $n \rightarrow \infty$.
- 8.9. If $y \in f(\mathbb{R})$ what is $f(y)$?
- 8.10. Consider the function $g(x) = f(x) - f(x + 1/999)$. Use the intermediate value theorem.
- 9.1. Divide the set into n subsets each of which has only pairwise relatively prime numbers.
- 9.2. Divide the set into n subsets each of which contains only numbers which are multiple or divisor of the other ones.
- 9.3. Look at pairs of numbers in that sequence whose sum is precisely 104. Those pairs may not cover the whole progression, but that can be fixed...
- 9.4. Prove that for each $k = 1, 2, \dots, 8$, at least 2 of the elements given coincide at place k . Consider a pair of elements which coincide at place 1, another pair of elements which coincide at place 2, and so on. How many pairs of elements do we have?
- 9.5. Consider the sequences a_i = number of games played from the 1st through the j th day of the month, and $b_j = a_j + 14$. Put them together and use the pigeonhole principle to prove that two elements must be the equal.
- 10.1. Try to re-write the n th term of the sum as $\frac{A_k}{3^k - 2^k} - \frac{B_k}{3^{k+1} - 2^{k+1}}$.

10.2. If you write a few terms of the product will notice a lot of cancellations. Factor the numerator and denominator of the n th term of the product and cancel all possible factors from $k = 2$ to $k = N$. You get an expression in N . Find its limit as $N \rightarrow \infty$.

11.1. Start by symmetrizing the given function:

$$f(x, y, z) = T(x, y, z) + T(y, z, x) + T(z, x, y).$$

11.2. Look at the expression $f(x) + f(\frac{\pi}{2} - x)$.

12.1. —

12.2. Find the number of distributions of ice-cream cones without the restriction "using all k flavors". Then remove the distributions in which at least one of the flavors is unused.

12.3. If P_i is the set of permutations fixing element a_i , then the set of non-derangements are the elements of the $P_1 \cup P_2 \cup \dots \cup P_n$.

13.1. Compare to $\frac{10^{20000} - 3^{200}}{10^{100} + 3}$.

13.2. If $\cos u$, $\sin u$, $\cos v$, and $\sin v$ are rational, so are $\cos u + v$ and $\sin u + v$.

13.3. Contradiction.

13.4. Since the values are positive integers, one of them must be the smallest one. What are the values of the neighbors of a square with minimum value?

13.5. Try punches at $(0, 0)$, $(\pm\alpha, 0)$, $\dots$ for some appropriate α .

13.6. Note that $\frac{f(n+2)-f(n+1)}{f(n+1)-f(n)}$ is a very simple polynomial in n .

13.7. Start by finding some subset T of S as large as possible and such that any three elements of it are pairwise relatively prime.

13.8. Try a binary operation that depends only on the first element: $x * y = \phi(x)$.

SOLUTIONS

- 1.1.** We prove it by induction. The basis step corresponds to $n = 4$, and in this case certainly we have $4! > 2^4$ ($24 > 16$). Next, for the induction step, assume the inequality holds for some value of $n \geq 4$, i.e., we assume $n! > 2^n$, and look at what happens for $n + 1$:

$$(n + 1)! = n! (n + 1) > 2^n (n + 1) > 2^n \cdot 2 = 2^{n+1}.$$

$\uparrow$
 by induction hypothesis

Hence the inequality also holds for $n + 1$. Consequently it holds for every $n \geq 4$.

- 1.2.** For the basis step, we have that for $n = 1$ indeed $2^{2 \cdot 1} - 1 = 4 - 1 = 3$ is divisible by 3. Next, for the inductive step, assume that $n \geq 1$ and $2^{2n} - 1$ is divisible by 3. We must prove that $2^{2(n+1)} - 1$ is also divisible by 3. We have

$$2^{2(n+1)} - 1 = 2^{2n+2} - 1 = 4 \cdot 2^{2n} - 1 = 3 \cdot 2^{2n} + (2^{2n} - 1).$$

In the last expression the last term is divisible by 3 by induction hypothesis, and the first term is also a multiple of 3, so the whole expression is divisible by 3 and we are done.

- 1.3.** By induction. For $n = 1$ we have that $a^1 - b^1 = a - b$ is indeed divisible by $a - b$. Next, for the inductive step, assume that $a^n - b^n$ is divisible by $a - b$. We must prove that $a^{n+1} - b^{n+1}$ is also divisible by $a - b$. In fact:

$$a^{n+1} - b^{n+1} = (a - b) a^n + b(a^n - b^n).$$

On the right hand side the first term is a multiple of $a - b$, and the second term is divisible by $a - b$ by induction hypothesis, so the whole expression is divisible by $a - b$.

- 1.4.** We prove it by strong induction. First we notice that the result is true for $n = 0$ ($F_0 = 0 < 1 = 2^0$), and $n = 1$ ($F_1 = 1 < 2 = 2^1$). Next, for the inductive step, assume that $n \geq 1$ and assume that the claim is true, i.e. $F_k < 2^k$, for every k such that $0 \leq k \leq n$. Then we must prove that the result is also true for $n + 1$. In fact:

$$F_{n+1} = F_n + F_{n-1} < 2^n + 2^{n-1} < 2^n + 2^n = 2^{n+1},$$

$\uparrow$
 by induction hypothesis

and we are done.

- 1.5.** We prove it by induction. For $n = 1$ the expression is indeed an integer. For $n = 2$ we have that $r^2 + 1/r^2 = (r + 1/r)^2 - 2$ is also an integer. Next assume that $n > 2$ and that the expression is an integer for $n - 1$ and n . Then we have

$$\left(r^{n+1} + \frac{1}{r^{n+1}}\right) = \left(r^n + \frac{1}{r^n}\right) \left(r + \frac{1}{r}\right) - \left(r^{n-1} + \frac{1}{r^{n-1}}\right),$$

hence the expression is also an integer for $n + 1$.

1.6. By experimentation we easily find:

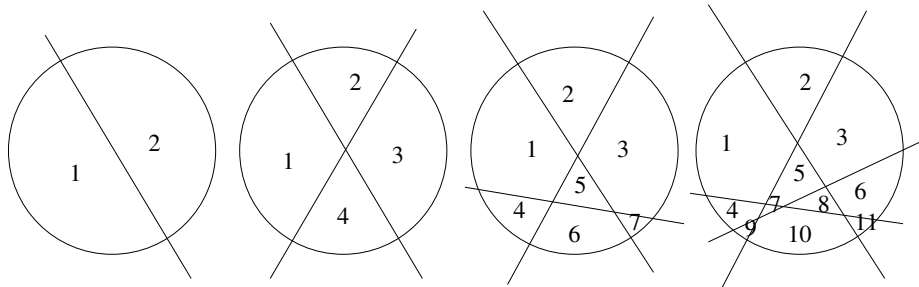


FIGURE 1. Plane regions.

n	1	2	3	4	...
$R(n)$	2	4	7	11	...

A formula that fits the first few cases is $R(n) = (n^2 + n + 2)/2$. We will prove by induction that it works for all $n \geq 1$. For $n = 1$ we have $R(1) = 2 = (1^2 + 1 + 2)/2$, which is correct. Next assume that the property is true for some positive integer n , i.e.:

$$R(n) = \frac{n^2 + n + 2}{2}.$$

We must prove that it is also true for $n + 1$, i.e.,

$$R(n + 1) = \frac{(n + 1)^2 + (n + 1) + 2}{2} = \frac{n^2 + 3n + 4}{2}.$$

So let's look at what happens when we introduce the $(n + 1)$ th straight line. In general this line will intersect the other n lines in n different intersection points, and it will be divided into $n + 1$ segments by those intersection points. Each of those $n + 1$ segments divides a previous region into two regions, so the number of regions increases by $n + 1$. Hence:

$$R(n + 1) = R(n) + n + 1.$$

But by induction hypothesis, $R(n) = (n^2 + n + 2)/2$, hence:

$$R(n + 1) = \frac{n^2 + n + 2}{2} + n + 1 = \frac{n^2 + 3n + 4}{2}.$$

QED.

1.7. We prove it by induction in the number n of lines. For $n = 1$ we will have two regions, and we can color them with just two colors, say one in red and the other one in blue. Next assume that the regions obtained after dividing the plane with n lines can always be colored with two colors, red and blue, so that no two regions that share a boundary have the same color. We need to prove that such kind of coloring is also possible after dividing the plane with $n + 1$ lines. So assume that the plane divided by n lines has been colored in the desired way. After we introduce the

$(n + 1)$ th line we need to recolor the plane to make sure that the new coloring still verifies that no two regions that share a boundary have the same color. We do it in the following way. The $(n + 1)$ th line divides the plane into two half-planes. We keep intact the colors in all the regions that lie in one half-plane, and reverse the colors (change red to blue and blue to red) in all the regions of the other half-plane. So if two regions share a boundary and both lie in the same half-plane, they will still have different colors. Otherwise, if they share a boundary but are in opposite half-planes, then they are separated by the $(n + 1)$ th line; which means they were part of the same region, so had the same color, and must have acquired different colors after recoloring.

1.8. The answer is $f(n) = n^2 - n + 2$. The proof is by induction. For $n = 1$ we get $f(1) = 2$, which is indeed correct. Then we must prove that if $f(n) = n^2 - n + 2$ then $f(n + 1) = (n + 1)^2 - (n + 1) + 2$. In fact, the $(n + 1)$ th great circle meets each of the other great circles in two points each, so $2n$ points in total, which divide the circle into $2n$ arcs. Each of these arcs divides a region into two, so the number of regions grow by $2n$ after introducing the $(n + 1)$ th circle. Consequently $f(n + 1) = n^2 - n + 2 + 2n = n^2 + n + 2 = (n + 1)^2 - (n + 1) + 2$, QED.

1.9. We proceed by induction. For the basis step, i.e. $n = 44$, we can use four 5¢ stamps and two 12¢ stamps, so that $5 \cdot 4 + 12 \cdot 2 = 44$. Next, for the induction step, assume that for a given $n \geq 44$ the task is possible by using x 5¢ stamps and y 12¢ stamps, i.e. $n = 5x + 12y$. We must now prove that we can find some combination of x' 5¢ stamps and y' 12¢ stamps so that $n + 1 = 5x' + 12y'$. First note that either $x \geq 7$ or $y \geq 2$ — otherwise we would have $x \leq 6$ and $y \leq 1$, hence $n \leq 5 \cdot 6 + 12 \cdot 1 = 42 < 44$, contradicting the hypothesis that $n \geq 44$. So we consider the two cases:

1. If $x \geq 7$, then we can accomplish the goal by setting $x' = x - 7$ and $y' = y + 6$:

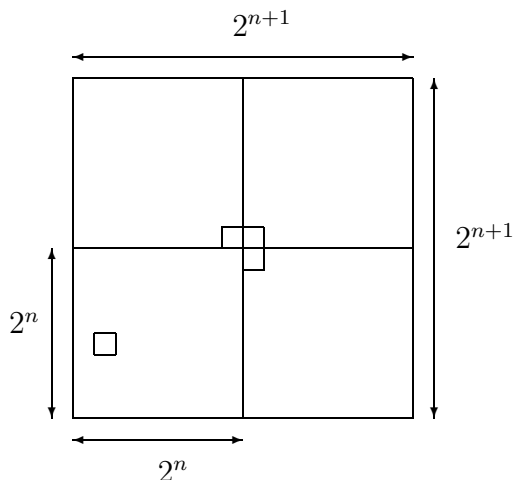
$$5x' + 12y' = 5(x - 7) + 12(y + 6) = 5x + 12y + 1 = n + 1.$$

2. On the other hand, if $y \geq 2$ then, we can do it by setting $x' = x + 5$ and $y' = y - 2$:

$$5x' + 12y' = 5(x + 5) + 12(y - 2) = 5x + 12y + 1 = n + 1.$$

1.10. We prove it by induction on n . For $n = 1$ the defective chessboard consists of just a single L and the tiling is trivial. Next, for the inductive step, assume that a $2^n \times 2^n$ defective chessboard can be tiled with L's. Now, given a $2^{n+1} \times 2^{n+1}$ defective chessboard, we can divide it into four $2^n \times 2^n$ chessboards as shown in the figure. One of them will have a square missing and will be defective, so it can be tiled with L's. Then we place an L covering exactly one corner of each of the other $2^n \times 2^n$ chessboards (see figure). The remaining part of each of those chessboards is like a defective chessboard and can be tiled in the desired way too. So the whole $2^{n+1} \times 2^{n+1}$ defective chessboard can be tiled with L's.

1.11. We use induction on the number n of piles. For $n = 1$ we have only one pile, and since each player must take at least one token from that pile, the number of tokens in

FIGURE 2. A $2^{n+1} \times 2^{n+1}$ defective chessboard.

it will decrease at each move until it is empty. Next, for the induction step, assume that the game with n piles must end eventually. We will prove that the same is true for $n + 1$ piles. First note that the players cannot keep taking tokens only from the first n piles, since by induction hypothesis the game with n piles eventually ends. So sooner or later one player must take a token from the $(n + 1)$ th pile. It does not matter how many tokens he or she adds to the other n piles after that, it is still true that the players cannot keep taking tokens only from the first n piles forever, so eventually someone will take another token from the $(n + 1)$ th pile. Consequently, the number of tokens in that pile will continue decreasing until it is empty. After that we will have only n piles left, and by induction hypothesis the game will end in finitely many steps after that.¹

1.12. The numbers 8 and 9 are a pair of consecutive square-fulls. Next, if n and $n + 1$ are square-full, so are $4n(n + 1)$ and $4n(n + 1) + 1 = (2n + 1)^2$.

1.13. For $n = 2, 3, 4, 5, 6$ we have:

$$\begin{aligned} (1 + x + x^2)^2 &= 1 + 2x + 3x^2 + 2x^3 + x^4 \\ (1 + x + x^2)^3 &= 1 + 3x + 6x^2 + 7x^3 + \dots \\ (1 + x + x^2)^4 &= 1 + 4x + 10x^2 + 16x^3 + \dots \\ (1 + x + x^2)^5 &= 1 + 5x + 15x^2 + 30x^3 + \dots \\ (1 + x + x^2)^6 &= 1 + 6x + 21x^2 + 50x^3 + \dots \end{aligned}$$

¹An alternate proof based on properties of ordinal numbers is as follows (requires some advanced set-theoretical knowledge.) Here ω = first infinite ordinal number, i.e., the first ordinal after the sequence of natural numbers $0, 1, 2, 3, \dots$. Let the ordinal number $\alpha = a_0 + a_1\omega + a_2\omega^2 + \dots + a_{n-1}\omega^{n-1}$ represent a configuration of n piles with $a_0, a_1, \dots, a_{n-1}$ tokens respectively (read from left to right.) After a move the ordinal number representing the configuration of tokens always decreases. Every decreasing sequence of ordinal numbers is finite. Hence the result.

In general, if $(1 + x + x^2)^n = a + bx + cx^2 + dx^3 + \dots$, then

$$(1 + x + x^2)^{n+1} = a + (a + b)x + (a + b + c)x^2 + (b + c + d)x^3 + \dots,$$

hence the first four coefficients of $(1 + x + x^2)^{n+1}$ depend only on the first four coefficients of $(1 + x + x^2)^n$. The same is true if we write the coefficients modulo 2, i.e., as “0” if they are even, or “1” if they are odd. So, if we call $q_n(x) = (1 + x + x^2)^n$ with the coefficients written modulo 2, we have

$$\begin{aligned} q_1(x) &= 1 + 1x + 1x^2 \\ q_2(x) &= 1 + 0x + 1x^2 + 0x^3 + 1x^4 \\ q_3(x) &= 1 + 1x + 0x^2 + 1x^3 + \dots \\ q_4(x) &= 1 + 0x + 0x^2 + 0x^3 + \dots \\ q_5(x) &= 1 + 1x + 1x^2 + 0x^3 + \dots \\ q_6(x) &= 1 + 0x + 1x^2 + 0x^3 + \dots \end{aligned}$$

We notice that the first four coefficients of $q_6(x)$ coincide with those of $q_2(x)$, and since these first four coefficients determine the first four coefficients of each subsequent polynomial of the sequence, they will repeat periodically so that those of $q_n(x)$ will always coincide with those of q_{n+4} . Since for $n = 2, 3, 4, 5$ at least one of the first four coefficients of $q_n(x)$ is 0 (equivalently, at least one of the first four coefficients of $(1 + x + x^2)^n$ is even), the same will hold for all subsequent values of n .

2.1. Using the AGM Inequality on each factor of the LHS we get

$$\left(\frac{a^2b + b^2c + c^2a}{3} \right) \left(\frac{ab^2 + bc^2 + ca^2}{3} \right) \geq \left(\sqrt[3]{a^3b^3c^3} \right) \left(\sqrt[3]{a^3b^3c^3} \right) = a^2b^2c^2.$$

Multiplying by 9 we get the desired inequality.

Another solution consists of using the Cauchy-Schwarz inequality:

$$\begin{aligned} (a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) &= \\ &= \left((a\sqrt{b})^2 + (b\sqrt{c})^2 + (c\sqrt{a})^2 \right) \left((\sqrt{b}c)^2 + (\sqrt{c}a)^2 + (\sqrt{a}b)^2 \right) \\ &\geq (abc + abc + abc)^2 \\ &= 9a^2b^2c^2. \end{aligned}$$

2.2. This result is the AGM applied to the set of numbers $1, 2, \dots, n$:

$$\sqrt[n]{1 \cdot 2 \cdot \dots \cdot n} < \frac{1 + 2 + \dots + n}{n} = \frac{\frac{n(n+1)}{2}}{n} = \frac{n+1}{2}.$$

Raising both sides to the n th power we get the desired result.

2.3. The simplest solution consists of using the weighted power means inequality to the (weighted) arithmetic and quadratic means of x and y with weights $\frac{p}{p+q}$ and $\frac{q}{p+q}$:

$$\frac{p}{p+q}x + \frac{q}{p+q}y \leq \sqrt{\frac{p}{p+q}x^2 + \frac{q}{p+q}y^2},$$

hence

$$(px + qy)^2 \leq (p+q)(px^2 + qy^2).$$

Or we can use the Cauchy-Schwarz inequality as follows:

$$\begin{aligned} (px + qy)^2 &= (\sqrt{p}\sqrt{p}x + \sqrt{q}\sqrt{q}y)^2 \\ &\leq (\{\sqrt{p}\}^2 + \{\sqrt{q}\}^2) (\{\sqrt{p}x\}^2 + \{\sqrt{q}y\}^2) \quad (\text{Cauchy-Schwarz}) \\ &= (p+q)(px^2 + qy^2). \end{aligned}$$

Finally we use $p+q \leq 1$ to obtain the desired result.

2.4. By the power means inequality:

$$\underbrace{\frac{a+b+c}{3}}_{M^1(a,b,c)} \geq \underbrace{\left(\frac{\sqrt{a} + \sqrt{b} + \sqrt{c}}{3}\right)^2}_{M^{1/2}(a,b,c)}$$

From here the desired result follows.

2.5. We have:

$$\begin{aligned} x + y + z &= (x + y + z)\sqrt[3]{xyz} && (xyz = 1) \\ &\leq \frac{(x + y + z)^2}{3} && (\text{AGM inequality}) \\ &\leq x^2 + y^2 + z^2. && (\text{power means inequality}) \end{aligned}$$

2.6. The result can be obtained by using Minkowski's inequality repeatedly:

$$\begin{aligned} \sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2} + \cdots + \sqrt{a_n^2 + b_n^2} &\geq \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2} + \cdots + \sqrt{a_n^2 + b_n^2} \\ &\geq \sqrt{(a_1 + a_2 + a_3)^2 + (b_1 + b_2 + b_3)^2} + \cdots \\ &\quad + \sqrt{a_n^2 + b_n^2} \\ &\quad \dots \\ &\geq \sqrt{(a_1 + a_2 + \cdots + a_n)^2 + (b_1 + b_2 + \cdots + b_n)^2} \end{aligned}$$

Another way to think about it is geometrically. Consider a sequence of points in the plane $P_k = (x_k, y_k)$, $k = 0, \dots, n$, such that

$$(x_k, y_k) = (x_{k-1} + a_k, y_{k-1} + b_k) \quad \text{for } k = 1, \dots, n.$$

Then the left hand side of the inequality is the sum of the distances between two consecutive points, while the right hand side is the distance between the first one and the last one:

$$d(P_0, P_1) + d(P_1, P_2) + \cdots + d(P_{n-1}, P_n) \leq d(P_0, P_n).$$

2.7. By the Arithmetic-Geometric Mean Inequality

$$1 = \sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n},$$

Hence $f(x_1, x_2, \dots, x_n) \geq n$. On the other hand $f(1, 1, \dots, 1) = n$, so the minimum value is n .

2.8. For $x = y = z = 1$ we see that $S = 3/2$. We will prove that in fact $3/2$ is the minimum value of S by showing that $S \geq 3/2$.

Note that

$$S = \left(\frac{x}{\sqrt{y+z}} \right)^2 + \left(\frac{y}{\sqrt{z+x}} \right)^2 + \left(\frac{z}{\sqrt{x+y}} \right)^2.$$

Hence by the Cauchy-Schwarz inequality:

$$S \cdot (u^2 + v^2 + w^2) \geq \left(\frac{xu}{\sqrt{y+z}} + \frac{yv}{\sqrt{z+x}} + \frac{zw}{\sqrt{x+y}} \right)^2.$$

Writing $u = \sqrt{y+z}$, $v = \sqrt{z+x}$, $w = \sqrt{x+y}$ we get

$$S \cdot 2(x+y+z) \geq (x+y+z)^2,$$

hence, dividing by $2(x+y+z)$ and using the AGM inequality:

$$S \geq \frac{1}{2}(x+y+z) \geq \frac{1}{2} \cdot 3\sqrt[3]{xyz} = \frac{3}{2}.$$

2.9. By the Arithmetic-Harmonic Mean inequality:

$$\frac{3}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}} \leq \frac{x+y+z}{3} = \frac{1}{3},$$

hence

$$9 \leq \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

On the other hand for $x = y = z = 1/3$ the sum is 9, so the minimum value is 9.

2.10. Assume $a \leq b \leq c$, $A \leq B \leq C$. Then by Chebyshev's Inequality:

$$\frac{aA + bB + cC}{3} \geq \left(\frac{a+b+c}{3} \right) \left(\frac{A+B+C}{3} \right).$$

Using $A + B + C = \pi$ and dividing by $(a+b+c)/3$ we get the desired result.

- 2.11.** Assume $a_i + b_i > 0$ for each i (otherwise both sides are zero). Then by the AGM inequality

$$\left(\frac{a_1 \cdots a_n}{(a_1 + b_1) \cdots (a_n + b_n)} \right)^{1/n} \leq \frac{1}{n} \left(\frac{a_1}{a_1 + b_1} + \cdots + \frac{a_n}{a_n + b_n} \right)$$

and similarly with the roles of a and b reversed. Adding both inequalities and clearing denominators we get the desired result.

- 2.12.** We have that $n!$ is increasing for $n \geq 1$, i.e., $1 \leq n < m \implies n! < m!$. So $1999! > 2000 \implies (1999!)! > 2000! \implies ((1999!)!)! > (2000!)! \implies \dots \implies 1999!^{(2000)} > 2000!^{(1999)}$.

- 2.13.** Consider the function $f(x) = (1999 - x) \ln(1999 + x)$. Its derivative is $f'(x) = -\ln(1999 + x) + \frac{1999 - x}{1999 + x}$, which is negative for $0 \leq x \leq 1$, because in that interval

$$\frac{1999 - x}{1999 + x} \leq 1 = \ln e < \ln(1999 + x).$$

Hence f is decreasing in $[0, 1]$ and $f(0) > f(1)$, i.e., $1999 \ln 1999 > 1998 \ln 2000$. Consequently $1999^{1999} > 2000^{1998}$.

- 2.14.** We have $b^2 < \underbrace{b^2 + b + 1}_{a^2} < b^2 + 2b + 1 = (b + 1)^2$. But b^2 and $(b + 1)^2$ are consecutive squares, so there cannot be a square strictly between them.

- 2.15.** We may assume that the leading coefficient is $+1$. The sum of the squares of the roots of $x^n + a_1 x^{n-1} + \cdots + a_n$ is $a_1^2 - 2a_2$. The product of the squares of the roots is a_n^2 . Using the AGM inequality we have

$$\frac{a_1^2 - 2a_2}{n} \geq \sqrt[n]{a_n^2}.$$

Since the coefficients are ± 1 that inequality is $(1 \pm 2)/n \geq 1$, hence $n \leq 3$.

Remark: $x^3 - x^2 - x + 1 = (x + 1)(x - 1)^2$ is an example of 3th degree polynomial with all coefficients equal to ± 1 and only real roots.

- 2.16.** The given function is the square of the distance between a point of the quarter of circle $x^2 + y^2 = 2$ in the open first quadrant and a point of the half hyperbola $xy = 9$ in that quadrant. The tangents to the curves at $(1, 1)$ and $(3, 3)$ separate the curves, and both are perpendicular to $x = y$, so those points are at the minimum distance, and the answer is $(3 - 1)^2 + (3 - 1)^2 = 8$.

- 2.17.** Let

$$P = \left(\frac{1}{2}\right) \left(\frac{3}{4}\right) \cdots \left(\frac{2n-1}{2n}\right), \quad Q = \left(\frac{2}{3}\right) \left(\frac{4}{5}\right) \cdots \left(\frac{2n-2}{2n-1}\right).$$

We have $PQ = \frac{1}{2n}$. Also $\frac{1}{2} < \frac{2}{3} < \frac{3}{4} < \frac{4}{5} < \dots < \frac{2n-1}{2n}$, hence $2P \geq Q$, so $2P^2 \geq PQ = \frac{1}{2n}$, and from here we get $P \geq \frac{1}{\sqrt{4n}}$.

On the other hand we have $P < Q \frac{2n}{2n+1} < Q$, hence $P^2 < PQ = \frac{1}{2n}$, and from here $P < \frac{1}{\sqrt{2n}}$.

2.18. The given inequality is equivalent to

$$\frac{(m+n)!}{m!n!} m^m n^n = \binom{m+n}{n} m^m n^n < (m+n)^{m+n},$$

which is obviously true because the binomial expansion of $(m+n)^{m+n}$ includes the term on the left plus other terms.

2.19. Using the AGM inequality we get:

$$\frac{1}{n} \left\{ \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_n}{b_n} \right\} \geq \sqrt[n]{\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} \cdot \dots \cdot \frac{a_n}{b_n}} = 1.$$

From here the desired result follows.

2.20. We prove it by induction. For $n = 1$ the result is trivial, and for $n = 2$ it is a simple consequence of the following:

$$0 \leq (a_2 - a_1)(b_2 - b_1) = (a_1 b_1 + a_2 b_2) - (a_1 b_2 + a_2 b_1).$$

Next assume that the result is true for some $n \geq 2$. We will prove that it is true for $n + 1$. There are two possibilities:

1. If $x_{n+1} = b_{n+1}$, then we can apply the induction hypothesis to the n first terms of the sum and we are done.
2. If $x_{n+1} \neq b_{n+1}$, then $x_j = b_{n+1}$ for some $j \neq n + 1$, and $x_{n+1} = b_k$ for some $k \neq n + 1$. Hence:

$$\begin{aligned} \sum_{i=1}^{n+1} a_i x_i &= \sum_{\substack{i=1 \\ i \neq j}}^n a_i x_i + a_j x_j + a_{n+1} x_{n+1} \\ &= \sum_{\substack{i=1 \\ i \neq j}}^n a_i x_i + a_j b_{n+1} + a_{n+1} b_k \end{aligned}$$

(using the inequality for the two-term increasing sequences a_j, a_{n+1} and b_k, b_{n+1})

$$\leq \sum_{\substack{i=1 \\ i \neq j}}^n a_i x_i + a_j b_k + a_{n+1} b_{n+1}.$$

This reduces the problem to case 1.

- 3.1.** If p and q are consecutive primes and $p + q = 2r$, then $r = (p + q)/2$ and $p < r < q$, but there are no primes between p and q .
- 3.2.** Assume that the set of primes of the form $4n + 3$ is finite. Let P be their product. Consider the number $N = P^2 - 2$. Note that the square of an odd number is of the form $4n + 1$, hence P^2 is of the form $4n + 1$ and N will be of the form $4n + 3$. Now, if all prime factors of N were of the form $4n + 1$, N would be of the form $4n + 1$, so N must have some prime factor p of the form $4n + 3$. So it must be one of the primes in the product P , hence p divides $N - P^2 = 2$, which is impossible.
- 3.3.** That is equivalent to proving that $n^3 + 2n$ and $n^4 + 3n^2 + 1$ are relatively prime for every n . These are two possible ways to show it:
- Assume a prime p divides $n^3 + 2n = n(n^2 + 2)$. Then it must divide n or $n^2 + 2$. Writing $n^4 + 3n^2 + 1 = n^2(n^2 + 3) + 1 = (n^2 + 1)(n^2 + 2) - 1$ we see that p cannot divide $n^4 + 3n^2 + 1$ in either case.
 - The following identity

$$(n^2 + 1)(n^4 + 3n^2 + 1) - (n^3 + 2n)^2 = 1$$

(which can be checked algebraically) shows that any common factor of $n^4 + 3n^2 + 1$ and $n^3 + 2n$ should divide 1, so their gcd is always 1. (Note: if you are wondering how I arrived to that identity, I just used the Euclidean algorithm on the two given polynomials.)

- 3.4.** Assume $p(x) = a_0 + a_1x + \cdots + a_nx^n$, with $a_n \neq 0$. We will assume WLOG that $a_n > 0$, so that $p(k) > 0$ for every k large enough—otherwise we can use the argument below with $-p(x)$ instead of $p(x)$. We have

$$p(p(k) + k) = \sum_{i=0}^n a_i [p(k) + k]^i.$$

For each term of that sum we have that

$$a_i [p(k) + k]^i = [\text{multiple of } p(k)] + a_i k^i,$$

and the sum of the $a_i k^i$ is precisely $p(k)$, so $p(p(k) + k)$ is a multiple of $p(k)$. It remains only to note that $p(p(k) + k) \neq p(k)$ for infinitely many positive integers k , otherwise $p(p(x) + x)$ and $p(x)$ would be the same polynomial, which is easily ruled out for non constant $p(x)$.

- 3.5.** This can be proved easily by induction. Base case: $F_1 = 1$ and $F_2 = 1$ are in fact relatively prime. Induction Step: we must prove that if F_n and F_{n+1} are relatively prime then so are F_{n+1} and F_{n+2} . But this follows from the recursive definition of the Fibonacci sequence: $F_n + F_{n+1} = F_{n+2}$; any common factor of F_{n+1} and F_{n+2} would be also a factor of F_n , and consequently it would be a common factor of F_n and F_{n+1} (which by induction hypothesis are relatively prime.)

- 3.6.** For any integer n we have that n^2 only can be 0 or 1 mod 3. So if 3 does not divide a or b they must be 1 mod 3, and their sum will be 2 modulo 3, which cannot be a square.
- 3.7.** Assume the sum S is an integer. Let 2^i be the maximum power of 2 dividing n , and let 2^j be the maximum power of 2 dividing $n!$. Then

$$\frac{n!}{2^j} 2^i S = \sum_{k=1}^n \frac{n!}{k 2^{j-i}}.$$

For $n \geq 2$ the left hand side is an even number. In the right hand side all the terms of the sum are even integers except the one for $k = 2^i$ which is an odd integer, so the sum must be odd. Hence we have an even number equal to an odd number, which is impossible.

- 3.8.** Since each digit cannot be greater than 9, we have that $f(n) \leq 9 \cdot (1 + \log_{10} n)$, so in particular $f(N) \leq 9 \cdot (1 + 4444 \cdot \log_{10} 4444) < 9 \cdot (1 + 4444 \cdot 4) = 159993$. Next we have $f(f(N)) \leq 9 \cdot 6 = 54$. Finally among numbers not greater than 54, the one with the greatest sum of the digits is 49, hence $f(f(f(N))) \leq 4 + 9 = 13$. Next we use that $n \equiv f(n) \pmod{9}$. Since $4444 \equiv 7 \pmod{9}$, then

$$4444^{4444} \equiv 7^{4444} \pmod{9}.$$

We notice that the sequence $7^n \pmod{9}$ for $n = 0, 1, 2, \dots$ is $1, 7, 4, 1, 7, 4, \dots$, with period 3. Since $4444 \equiv 1 \pmod{3}$, we have $7^{4444} \equiv 7^1 \pmod{9}$, hence $f(f(f(N))) \equiv 7 \pmod{9}$. The only positive integer not greater than 13 that is congruent with 7 modulo 9 is 7, hence $f(f(f(N))) = 7$.

- 3.9.** Pick 1999 different prime numbers $p_1, p_2, \dots, p_{1999}$ (we can do that because the set of prime numbers is infinite) and solve the following system of 1999 congruences:

$$\begin{cases} x \equiv 0 & (\text{mod } p_1^3) \\ x \equiv -1 & (\text{mod } p_2^3) \\ x \equiv -2 & (\text{mod } p_3^3) \\ \dots & \\ x \equiv -1998 & (\text{mod } p_{1999}^3) \end{cases}$$

According to the Chinese Remainder Theorem, that system of congruences has a solution x (modulo $M = p_1^3 \dots p_{1999}^3$). For $k = 1, \dots, 1999$ we have that $x + k \equiv 0 \pmod{p_k^3}$, hence $x + k$ is in fact a multiple of p_k^3 .

- 3.10.** Assume $a \geq b \geq c$. Then

$$2 = \left(1 + \frac{1}{a}\right) \left(1 + \frac{1}{b}\right) \left(1 + \frac{1}{c}\right) \leq \left(1 + \frac{1}{c}\right)^3.$$

From here we get that $c < 4$, so its only possible values are $c = 1, 2, 3$.

For $c = 1$ we get $(1 + 1/c) = 2$, hence

$$\left(1 + \frac{1}{a}\right) \left(1 + \frac{1}{b}\right) = 1,$$

which is impossible.

For $c = 2$ we have $(1 + 1/c) = 3/2$, hence

$$\left(1 + \frac{1}{a}\right) \left(1 + \frac{1}{b}\right) = \frac{4}{3},$$

and from here we get

$$a = \frac{3(b+1)}{b-3},$$

with solutions $(a, b) = (15, 4)$, $(9, 5)$ and $(7, 6)$.

Finally for $c = 3$ we have $1 + 1/c = 1 + 1/3 = 4/3$, hence

$$\left(1 + \frac{1}{a}\right) \left(1 + \frac{1}{b}\right) = \frac{3}{2}.$$

So

$$a = \frac{2(b+1)}{b-2}.$$

The solutions are $(a, b) = (8, 3)$ and $(5, 4)$.

So the complete set of solutions verifying $a \geq b \geq c$ are

$$(a, b, c) = (15, 4, 2), (9, 5, 2), (7, 6, 2), (8, 3, 3), (5, 4, 3).$$

The rest of the triples verifying the given equation can be obtained by permutations of a, b, c .

- 3.11.** The change of variables $x = a + 1$, $y = b + 1$, $z = c + 1$, transforms the equation into the following one:

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1.$$

Assuming $x \leq y \leq z$ we have that $x \leq 3 \leq z$.

For $x = 1$ the equation becomes

$$\frac{1}{y} + \frac{1}{z} = 0.$$

which is impossible.

For $x = 2$ we have

$$\frac{1}{y} + \frac{1}{z} = \frac{1}{2},$$

or

$$z = \frac{2y}{y-2},$$

with solutions $(y, z) = (3, 6)$ and $(4, 4)$.

For $x = 3$ the only possibility is $(y, z) = (3, 3)$.

So the list of solutions is

$$(x, y, z) = (2, 3, 6), (2, 4, 4), (3, 3, 3),$$

and the ones obtained by permuting x, y, z .

With the original variables the solutions are (except for permutations of variables);

$$(a, b, c) = (1, 2, 5), (1, 3, 3), (2, 2, 2).$$

3.12. We look at the equation modulo 16. First we notice that $n^4 \equiv 0$ or $1 \pmod{16}$ depending on whether n is even or odd. On the other hand $1599 \equiv 15 \pmod{16}$. So the equation can be satisfied only if the number of odd terms in the LHS is 15 modulo 16, but that is impossible because there are only 14 terms in the LHS. Hence the equation has no solution.

3.13. Call $N = 10^{10^{10}}$, and consider the sequence $a_n = \text{remainder of dividing } F_n \text{ by } N$. Since there are only N^2 pairs of non-negative integers less than N , there must be two identical pairs $(a_i, a_{i+1}) = (a_j, a_{j+1})$ for some $0 \leq i < j$. Let $k = j - i$. Since $a_{n+2} = a_{n+1} + a_n$ and $a_{n-1} = a_{n+1} - a_n$, by induction we get that $a_n = a_{n+k}$ for every $n \geq 0$, so in particular $a_k = a_0 = 0$, and this implies that F_k is a multiple of N . (In fact since there are $N^2 + 1$ pairs (a_i, a_{i+1}) , for $i = 0, 1, \dots, N^2$, we can add the restriction $0 \leq i < j \leq N$ above and get that the result is true for some k such that $0 < k \leq N^2$.)

4.1. If $x = \sqrt{2} + \sqrt{5}$ then

$$\begin{aligned} x^2 &= 7 + 2\sqrt{10}, \\ x^2 - 7 &= 2\sqrt{10}, \\ (x^2 - 7)^2 &= 40, \\ x^4 - 14x^2 + 9 &= 0. \end{aligned}$$

Hence the desired polynomial is $x^4 - 14x^2 + 9$.

4.2. We prove it by showing that the sum is the root of a monic polynomial but not an integer—so by the rational roots theorem it must be irrational.

First we notice that $n < \sqrt{n^2 + 1} < n + 1/n$, hence the given sum is of the form

$$S = 1001 + \theta_1 + 1002 + \theta_2 + \cdots + 2000 + \theta_{1000}$$

were $0 < \theta_i < 1/1001$, consequently

$$0 < \theta_1 + \theta_2 + \cdots + \theta_{1000} < 1,$$

so S is not an integer.

Now we must prove that S is the root of a monic polynomial. More generally we will prove that a sum of the form

$$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n}$$

where the a_i 's are positive integers, is the root of a monic polynomial.² This can be proved by induction on n . For $n = 1$, $\sqrt{a_1}$ is the root of the monic polynomial $x^2 - a_1$. Next assume that $y = \sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n}$ is a zero of a monic polynomial $P(x) = x^r + c_{r-1}x^{r-1} + \cdots + c_0$. We will find a polynomial that has $z = y + \sqrt{a_{n+1}}$ as a zero. We have

$$0 = P(y) = P(z - \sqrt{a_{n+1}}) = (z - \sqrt{a_{n+1}})^r + c_{r-1}(z - \sqrt{a_{n+1}})^{r-1} + \cdots + c_0.$$

Expanding the parentheses and grouping the terms that contain $\sqrt{a_{n+1}}$:

$$0 = P(z - \sqrt{a_{n+1}}) = z^r + Q(z) + \sqrt{a_{n+1}} R(z).$$

Putting radicals on one side and squaring

$$(z^r + Q(z))^2 = a_{n+1} (R(z))^2,$$

so

$$T(x) = (x^r + Q(x))^2 - a_{n+1} (R(x))^2$$

is a monic polynomial with z as a root.

4.3. Consider the following polynomial:

$$Q(x) = (x+1)P(x) - x.$$

We have that $Q(k) = 0$ for $k = 0, 1, 2, \dots, n$, hence, by the *Factor theorem*,

$$Q(x) = Cx(x-1)(x-2)\cdots(x-n),$$

where C is a constant to be determined. Plugging $x = -1$ we get

$$Q(-1) = C(-1)(-2)\cdots(-(n+1)).$$

On the other hand $Q(-1) = 0 \cdot P(-1) - (-1) = 1$, hence $C = \frac{(-1)^{n+1}}{(n+1)!}$.

Next, plugging in $x = n+1$ we get

$$(n+2)P(n+1) - (n+1) = C(n+1)! = \frac{(-1)^{n+1}}{(n+1)!}(n+1)! = (-1)^{n+1},$$

hence

$$\boxed{P(n+1) = \frac{n+1+(-1)^{n+1}}{n+2}}.$$

²The result can be obtained also by resorting to a known theorem on *algebraic integers* ("algebraic integer" is the mathematical term used to designate a root of a monic polynomial.) It is known that algebraic integers form a mathematical structure called *ring*, basically meaning that the sum, difference and product of two algebraic integers is an algebraic integer. Now, if a_i and k_i are positive integers, then $\sqrt[k_i]{a_i}$ is an algebraic integer, because it is a root of the monic polynomial $x^{k_i} - a_i$. Next, since the sum or difference of algebraic integers is an algebraic integer then $\pm \sqrt[k_1]{a_1} \pm \sqrt[k_2]{a_2} \pm \cdots \pm \sqrt[k_n]{a_n}$ is in fact an algebraic integer (note that the roots do not need to be square roots, and the signs can be combined in any way.)

4.4. Let the zeros be a, b, c, d . The relationship between zeros and coefficients yields

$$\begin{aligned} a + b + c + d &= 18 \\ ab + ac + ad + bc + bd + cd &= k \\ abc + abd + acd + bcd &= -200 \\ abcd &= -1984. \end{aligned}$$

Assume $ab = -32$ and let $u = a + b$, $v = c + d$, $w = cd$. Then

$$\begin{aligned} u + v &= 18 \\ -32 + uv + w &= k \\ -32v + uw &= -200 \\ -32w &= -1984. \end{aligned}$$

From the last equation we get $w = 62$, and replacing in the other equations we easily get $u = 4$, $v = 14$. Hence

$$k = -32 + 4 \cdot 14 + 62 = \boxed{86}.$$

4.5. If k is an even integer we have $p(k) \equiv p(0) \pmod{2}$, and if it is odd then $p(k) \equiv p(1) \pmod{2}$. Since $p(0)$ and $p(1)$ are odd we have $p(k) \equiv 1 \pmod{2}$ for every integer k , so $p(k)$ cannot be zero.

4.6. We must prove that $P(1) = 0$. Consider the four complex numbers $\rho_k = e^{2\pi i k/5}$, $k = 1, 2, 3, 4$. All of them verify $\rho_k^5 = 1$, so together with 1 they are the roots of $x^5 - 1$. Since $x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1)$ then the ρ_k 's are the roots of $x^4 + x^3 + x^2 + x + 1$. So

$$P(1) + \rho_k Q(1) + \rho_k^2 R(1) = 0.$$

Adding for $k = 1, 2, 3, 4$ and taking into account that the numbers ρ_k^2 are the ρ_k 's in a different order we get

$$P(1) = 0.$$

4.7. The answer is no. We prove it by contradiction. Assume $(x - a)(x - b)(x - c) - 1 = p(x)q(x)$, where p is linear and q is quadratic. Then $p(a)q(a) = p(b)q(b) = p(c)q(c) = -1$. If the coefficients of p and q must integers they can take only integer values, so in each product one of the factor must be 1 and the other one is -1 . Hence either $p(x)$ takes the value 1 twice or it takes the value -1 twice. But a 1st degree polynomial cannot take the same value twice.

4.8. We prove it by contradiction. Suppose $g(x) = h(x)k(x)$, where $h(x)$ and $k(x)$ are non-constant polynomials with integral coefficients. Since $g(x) > 0$ for every x , $h(x)$ and $k(x)$ cannot have real roots, so they cannot change signs and we may suppose $h(x) > 0$ and $k(x) > 0$ for every x . Since $g(p_i) = 1$ for $i = 1, \dots, n$, we have $h(p_i) = k(p_i) = 1$, $i = 1, \dots, n$. If either $h(x)$ or $k(x)$ had degree less than n , it would constant, against the hypothesis, so they must be of degree n . Also we

know that $h(x) - 1$ and $k(x) - 1$ are zero for $x = p_i$, $i = 1, \dots, n$, so their roots are precisely $p_1, \dots, p_n$, and we can write

$$h(x) = 1 + a(x - p_1) \cdots (x - p_n)$$

$$k(x) = 1 + b(x - p_1) \cdots (x - p_n),$$

where a and b are integers. So we have

$$\begin{aligned} (x - p_1)^2 \cdots (x - p_n)^2 + 1 = \\ 1 + (a + b)(x - p_1) \cdots (x - p_n) + ab(x - p_1)^2 \cdots (x - p_n)^2. \end{aligned}$$

Hence

$$\begin{cases} a + b &= 0 \\ ab &= 1, \end{cases}$$

which is impossible, because there are no integers a, b verifying those equations.

4.9. Assume the quotient is $q(x)$ and the remainder is $r(x) = ax^2 + bx + c$. Then

$$x^{81} + x^{49} + x^{25} + x^9 + x = q(x)(x^3 - x) + r(x),$$

Plugging in the values $x = -1, 0, 1$ we get $r(-1) = -5$, $r(0) = 0$, $r(1) = 5$. From here we get $a = c = 0$, $b = 5$, hence the remainder is $r(x) = 5x$.

5.1. If $m = a^2 + b^2$ and $n = c^2 + d^2$, then consider the product $z = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$. We have

$$|z|^2 = |a + bi|^2 |c + di|^2 = (a^2 + b^2)(c^2 + d^2) = mn,$$

and

$$|z|^2 = (ac - bd)^2 + (ad + bc)^2,$$

so mn is also in fact a sum of two perfect squares.

5.2. The left hand side of the equality is the imaginary part of

$$\sum_{k=0}^n e^{ik} = \frac{e^{i(n+1)} - 1}{e^i - 1} = \frac{e^{i(n+1/2)} - e^{i/2}}{e^{i/2} - e^{-i/2}} = \frac{\cos(n + \frac{1}{2}) - \cos \frac{1}{2} + i\{\sin(n + \frac{1}{2}) - \sin \frac{1}{2}\}}{2i \sin \frac{1}{2}}.$$

The imaginary part of that expression is

$$\frac{\cos \frac{1}{2} - \cos(n + \frac{1}{2})}{2 \sin \frac{1}{2}} = \frac{\sin \frac{n}{2} \sin \frac{n+1}{2}}{\sin \frac{1}{2}}$$

5.3. Letting $x = e^{ia}$ we have that $x + 1/x = e^{ia} + e^{-ia} = 2 \cos a$, hence:

$$x^n + 1/x^n = e^{ina} + e^{-ina} = 2 \cos na.$$

5.4. Factoring a polynomial is easier to accomplish if we can find its roots. In this case we will look for roots that are roots of unity $e^{2k\pi i/n}$:

$$p(e^{2k\pi i/n}) = e^{10k\pi i/n} + e^{2k\pi i/n} + 1.$$

The three terms of that expression are complex numbers placed on the unit circle at the vertices of an equilateral triangle for $n = 3$ and $k = 1, 2$, so if $\omega = e^{2k\pi i/3}$, then

ω and ω^2 are roots of $p(z)$, hence $p(z)$ is divisible by $(z - \omega)(z - \omega^2) = z^2 + z + 1$. By long division we find that the other factor is $z^3 - z^2 + 1$, hence:

$$p(z) = (z^2 + z + 1)(z^3 - z^2 + 1).$$

5.5. Write $\sin t = (e^{ti} - e^{-ti})/2i$ and consider the polynomial

$$p(x) = \prod_{k=1}^{n-1} (x - e^{2\pi i k/n}).$$

We have:

$$P = \prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \prod_{k=1}^{n-1} \frac{e^{\pi i k/n} - e^{-\pi i k/n}}{2i} = \frac{e^{-\pi i (n-1)/2}}{(2i)^{n-1}} \prod_{k=1}^{n-1} (e^{2\pi i k/n} - 1) = \frac{p(1)}{2^{n-1}}.$$

On the other hand the roots of $p(x)$ are all n th roots of 1 except 1, so $(x-1)p(x) = x^n - 1$, and

$$p(x) = \frac{x^n - 1}{x - 1} = 1 + x + x^2 + \cdots + x^{n-1}.$$

Consequently $p(1) = n$, and $P = \frac{n}{2^{n-1}}$.

5.6. Assume the vertices of the n -gon placed on the complex plane at the n th roots of unity $1, \zeta, \zeta^2, \dots, \zeta^{n-1}$, where $\zeta = e^{2\pi i/n}$. Then the length of the diagonal connecting vertices j and k is $|\zeta^j - \zeta^k|$, and the desired product can be written

$$P = \prod_{0 \leq j < k < n} |\zeta^j - \zeta^k|.$$

By symmetry we obtain the same product if we replace the condition $j < k$ with $k < j$, and multiplying both expressions together we get:

$$P^2 = \prod_{\substack{0 \leq j, k < n \\ j \neq k}} |\zeta^j - \zeta^k| = |\zeta^j| \prod_{\substack{0 \leq j, k < n \\ j \neq k}} |1 - \zeta^{k-j}|.$$

Note that $|\zeta^j| = 1$, and for each k , $r = k - j$ takes all non-zero values from $k - n + 1$ to k . Since $\zeta^r = \zeta^{r+n}$ we may assume that r ranges from 1 to $n - 1$, so we can rewrite the product like this:

$$P^2 = \left(\prod_{r=1}^{n-1} |1 - \zeta^r| \right)^n.$$

Next consider the polynomial

$$p(x) = \prod_{r=1}^{n-1} (x - \zeta^r).$$

Its roots are the same roots of $x^n - 1$ except 1, hence $x^n - 1 = (x - 1)p(x)$ and

$$p(x) = \frac{x^n - 1}{x - 1} = 1 + x + x^2 + \cdots + x^{n-1},$$

hence

$$\prod_{r=1}^{n-1} (1 - \zeta^r) = p(1) = n.$$

consequently $P^2 = n^n$, and $P = n^{n/2}$.

- 5.7.** Define $h(x) = f(x) + ig(x)$. Then h is differentiable and $h'(0) = bi$ for some $b \in \mathbb{R}$. The given equations can be reinterpreted as $h(x+y) = h(x)h(y)$. Differentiating respect to y and substituting $y = 0$ we get $h'(x) = h(x)h'(0) = bi \cdot h(x)$, so $h(x) = Ce^{bix}$ for some $C \in \mathbb{C}$. From $h(0+0) = h(0)h(0)$ we get $C = C^2$. If $C = 0$ then $h = 0$ and f and g would be constant, contradicting the hypothesis. Thus $C = 1$. Finally, for any $x \in \mathbb{R}$,

$$f(x)^2 + g(x)^2 = |h(x)|^2 = |e^{bix}|^2 = 1.$$

- 5.8.** Assume the lights placed on the complex plane at the n th roots of unity $1, \zeta, \zeta^2, \dots, \zeta^{n-1}$, where $\zeta = e^{2\pi i/n}$. Without loss of generality we may assume that the light at 1 is initially on. Now, if $d < n$ is a divisor of n and the lights $\zeta^a, \zeta^{a+d}, \zeta^{a+2d}, \dots, \zeta^{a+(\frac{n}{d}-1)d}$ have the same state, then we can change the state of this n/d lights. The sum of these is

$$\zeta^a + \zeta^{a+d} + \zeta^{a+2d} + \dots + \zeta^{a+(\frac{n}{d}-1)d} = \zeta^a \left(\frac{1 - \zeta^n}{1 - \zeta^d} \right) = \zeta^a \left(\frac{1 - 1}{1 - \zeta^d} \right) = 0.$$

So if we add up all the roots that are “on”, the sum will never change. The original sum was 1, and the goal is to get all the lights turned on. That sum will be

$$1 + \zeta + \zeta^2 + \dots + \zeta^{n-1} = \frac{1 - \zeta^n}{1 - \zeta} = 0 \neq 1.$$

Hence we can never turn on all the lights.

- 6.1.** We have

$$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n = (1+x)^n.$$

Differentiating respect to x :

$$\binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \dots + n\binom{n}{n}x^{n-1} = n(1+x)^{n-1}.$$

Plugging in $x = 1$ we get the desired identity

- 6.2.** The desired expression states the equality between the coefficient of x^n in each of the following expansions:

$$(1+x)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} x^k,$$

and

$$\{(1+x)^n\}^2 = \left\{ \sum_{k=0}^n \binom{n}{k} x^k \right\}^2 = \sum_{k=0}^n \sum_{i+j=k} \binom{n}{i} \binom{n}{j} x^k.$$

Taking into account that $\binom{n}{j} = \binom{n}{n-j}$, for $k = n$ we get

$$\sum_{i+j=n} \binom{n}{i} \binom{n}{j} = \sum_{i+j=n} \binom{n}{i} \binom{n}{n-j} = \sum_{i=1}^n \binom{n}{i}^2,$$

and that must be equal to the coefficient of x^n in $(1+x)^{2n}$, which is $\binom{2n}{n}$.

6.3. This is just a generalization of the previous problem. We have

$$(1+x)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} x^k,$$

and

$$\begin{aligned} (1+x)^m (1+x)^n &= \left\{ \sum_{i=0}^m \binom{m}{i} x^i \right\} \left\{ \sum_{j=0}^n \binom{n}{j} x^j \right\} \\ &= \sum_{k=0}^{m+n} \sum_{\substack{i+j=k \\ 0 \leq i, j \leq k}} \binom{n}{j} \binom{m}{i} x^k. \end{aligned}$$

The coefficient of x^k must be the same on both sides, so:

$$\binom{m+n}{k} = \sum_{\substack{i+j=k \\ 0 \leq i, j \leq k}} \binom{n}{j} \binom{m}{i} = \sum_{j=0}^k \binom{n}{j} \binom{m}{k-j},$$

where we replace $i = k - j$ in the last step.

6.4. The generating function for the Fibonacci sequence is

$$0 + x + x^2 + 2x^3 + 3x^4 + 5x^5 + \cdots = \frac{x}{1 - x - x^2}.$$

The desired sum is the left hand side with $x = 1/2$, hence its value is

$$0 + \frac{1}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \frac{5}{2^5} + \cdots = \frac{\frac{1}{2}}{1 - \frac{1}{2} - \frac{1}{2^2}} = \boxed{2}.$$

6.5. Let's look first at the general case $2a + 5b = n$. Let u_n be the number of nonnegative solutions to this equation. We need to find u_{100} . If we define

$$\begin{aligned} A(x) &= 1 + x^2 + x^4 + x^6 + \cdots \\ B(x) &= 1 + x^5 + x^{10} + x^{15} + \cdots \end{aligned}$$

we easily see that the generating function $f(x) = u_0 + u_1x + u_2x^2 + \cdots$ of the sequence u_n is

$$f(x) = A(x)B(x) = (1 + x^2 + x^4 + x^6 + \cdots)(1 + x^5 + x^{10} + x^{15} + \cdots).$$

In order to find the 100th coefficient we start by writing $A(x) = 1/(1 - x^2)$, $B(x) = 1/(1 - x^5)$, so:

$$f(x) = A(x)B(x) = \frac{1}{1 - x^2} \frac{1}{1 - x^5} = \frac{1}{1 - x^2 - x^5 + x^7}.$$

Hence

$$1 = (1 - x^2 - x^5 + x^7)(u_0 + u_1x + u_2x^2 + \cdots).$$

From here we get that $1 \cdot u_0 = 1$, hence $u_0 = 1$. Similarly $1 \cdot u_1 = 0$, hence $u_1 = 0$, etc., so we get $u_0 = u_2 = u_4 = u_5 = u_7 = 1$, and $u_1 = u_3 = 0$. Then for $k > 7$ the coefficient of x^k of the product must be

$$u_k - u_{k-2} - u_{k-5} + u_{k-7} = 0.$$

So we get the following recursive relation for the terms of the sequence:

$$u_k = u_{k-2} + u_{k-5} - u_{k-7}.$$

So now for instance we get $u_8 = u_6 + u_3 - u_1 = 1 + 0 - 0 = 1$, $u_9 = u_7 + u_4 - u_2 = 1 + 1 - 1 = 1$, and so on, and we get the following sequence:

1, 0, 1, 0, 1, 1, 1, 1, 1, 1, 2, 1, 2, 1, 2, 2, 2, 2, 2, 2, 3, 2, 3, 2, 3, 3, 3,
3, 3, 3, 4, 3, 4, 3, 4, 4, 4, 4, 4, 4, 5, 4, 5, 4, 5, 5, 5, 5, 5, 5, 6, 5, 6, 5,
6, 6, 6, 6, 6, 6, 7, 6, 7, 6, 7, 7, 7, 7, 7, 7, 8, 7, 8, 7, 8, 8, 8, 8, 8, 8, 9,
8, 9, 8, 9, 9, 9, 9, 9, 9, 10, 9, 10, 9, 10, 10, 10, 10, 10, 10, 11, ...

where $u_{100} = 11$.

6.6. The answer equals the coefficient of x^{10} in the expansion of

$$(1 + x + x^2 + \cdots + x^9)^6.$$

Since $1 + x + x^2 + \cdots = 1/(1 - x)$ the answer can be obtained also from the coefficient of x^{10} in the Maclaurin series of $1/(1 - x)^6 = (1 - x)^{-6}$. Since that includes six sequences of the form $0, 0, \dots, 10, \dots, 0$ we need to subtract 6, so the final answer is

$$\begin{aligned} \binom{-6}{10} - 6 &= \frac{(-6)(-7)(-8)(-9)(-10)(-11)(-12)(-13)(-14)(-15)}{10!} - 6 \\ &= 3003 - 6 = \boxed{2997}. \end{aligned}$$

6.7. Consider the polynomial $P(x) = a_1 + a_2x + a_3x^2 + \cdots + a_{50}x^{49}$. If r is a 3rd root of unity different from 1 then $P(r) = c(1 + r + r^2)$, where $c = a_k + a_{k+3} + a_{k+6} + \cdots$. But $1 + r + r^2 = (r^3 - 1)/(r - 1) = 0$, so $P(r) = 0$. Analogous reasoning shows that $P(r) = 0$ for each 5th, 7th, 11th, 13th, 17th root of unity r different from 1. Since there are respectively $2 + 4 + 6 + 10 + 12 + 16 = 50$ such roots of unity we have

that $P(r)$ is zero for 50 different values of r . But a 49-degree polynomial has only 49 roots, so $P(x)$ must be identically zero.

7.1. Let $f(n)$ be that number. Then we easily find $f(0) = 1$, $f(1) = 2$, $f(2) = 3$, $f(3) = 5, \dots$ suggesting that $f(n) = F_{n+2}$ (shifted Fibonacci sequence). We prove this by showing that $f(n)$ verifies the same recurrence as the Fibonacci sequence. The subsets of $\{1, 2, \dots, n\}$ that contain no two consecutive elements can be divided into two classes, the ones not containing n , and the ones containing n . The number of the ones not containing n is just $f(n-1)$. On the other hand the ones containing n cannot contain $n-1$, so their number equals $f(n-2)$. Hence $f(n) = f(n-1) + f(n-2)$, QED.

7.2. Let x_n be the number of regions in the plane determined by n “vee”s. Then $x_1 = 2$, and $x_{n+1} = x_n + 4n + 1$. We justify the recursion by noticing that the $(n+1)$ th “vee” intersects each of the other “vee”s at 4 points, so it is divided into $4n + 1$ pieces, and each piece divides one of the existing regions of the plane into two, increasing the total number of regions by $4n + 1$. So the answer is

$$x_n = 2 + (4 + 1) + (4 \cdot 2 + 1) + \dots + (4 \cdot (n-1) + 1) = 2n^2 - n + 1.$$

7.3. Let x_n be the number of tilings of an $n \times 2$ rectangle by dominoes. We easily find $x_1 = 1$, $x_2 = 2$. For $n \geq 3$ we can place the rightmost domino vertically and tile the rest of the rectangle in x_{n-1} ways, or we can place two horizontal dominoes to the right and tile the rest in x_{n-2} ways, so $x_n = x_{n-1} + x_{n-2}$. So the answer is the shifted Fibonacci sequence, $x_n = F_{n+1}$.

7.4. Let f_n denote the number of minimal selfish subsets of $\{1, 2, \dots, n\}$. For $n = 1$ we have that the only selfish set of $\{1\}$ is $\{1\}$, and it is minimal. For $n = 2$ we have two selfish sets, namely $\{1\}$ and $\{1, 2\}$, but only $\{1\}$ is minimal. So $f_1 = 1$ and $f_2 = 1$. For $n > 2$ the number of minimal selfish subsets of $\{1, 2, \dots, n\}$ not containing n is equal to f_{n-1} . On the other hand, for any minimal selfish set containing n , by removing n from the set and subtracting 1 from each remaining element we obtain a minimal selfish subset of $\{1, 2, \dots, n\}$. Conversely, any minimal selfish subset of $\{1, 2, \dots, n-2\}$ gives rise to a minimal selfish subset of $\{1, 2, \dots, n\}$ containing n by the inverse procedure. Hence the number of minimal selfish subsets of $\{1, 2, \dots, n\}$ containing n is f_{n-2} . Thus $f_n = f_{n-1} + f_{n-2}$, which together with $f_1 = f_2 = 1$ implies that $f_n = F_n$ (n th Fibonacci number.)

7.5. Assume that $b_1, b_2, \dots, b_n$ is a derangement of the sequence $a_1, a_2, \dots, a_n$. The element b_n can be any of $a_1, \dots, a_{n-1}$, so there are $n-1$ possibilities for its value. Once we have fixed the value of $b_n = a_k$ for some $k = 1, \dots, n-1$, the derangement can be of one of two classes: either $b_k = a_n$, or $b_k \neq a_n$. The first class coincides with the derangements of $a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_{n-1}$, and there are D_{n-2} of them. The second class coincides with the derangements of $a_1, \dots, a_{n-1}$ with a_k replaced with a_n , and there are D_{n-1} of them.

- 8.1.** The solution is based on the fact that $\sqrt{u^2} = |u|$. Letting $u = 1 \pm \sqrt{x-1}$ we have that $u^2 = x \pm 2\sqrt{x-1}$, hence the given function turns out to be:

$$f(x) = |1 + \sqrt{x-1}| + |1 - \sqrt{x-1}|,$$

Defined for $x \geq 1$.

The expression $1 + \sqrt{x-1}$ is always positive, hence $|1 + \sqrt{x-1}| = 1 + \sqrt{x-1}$. On the other hand $|1 - \sqrt{x-1}| = 1 - \sqrt{x-1}$ if $1 - \sqrt{x-1} \geq 0$ and $|1 - \sqrt{x-1}| = -1 + \sqrt{x-1}$ if $1 - \sqrt{x-1} \leq 0$, hence

$$f(x) = \begin{cases} 1 + \sqrt{x-1} + 1 - \sqrt{x-1} = 2 & \text{if } 1 - \sqrt{x-1} \geq 0 \\ 1 + \sqrt{x-1} - 1 + \sqrt{x-1} = 2\sqrt{x-1} & \text{if } 1 - \sqrt{x-1} < 0 \end{cases}$$

So the function is equal to 2 if $1 - \sqrt{x-1} \geq 0$, which happens for $1 \leq x \leq 2$. So $f(x) = 2$ (constant) in $[1, 2]$.

- 8.2.** The desired value is the limit of the following sequence:

$$\begin{aligned} a_1 &= \sqrt{2} \\ a_2 &= \sqrt{2 + \sqrt{2}} \\ a_3 &= \sqrt{2 + \sqrt{2 + \sqrt{2}}} \\ &\dots \end{aligned}$$

defined by the recursion $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$ ($n \geq 1$).

First we must prove that the given sequence has a limit. To that end we prove

1. The sequence is bounded. More specifically, $0 < a_n < 2$ for every $n = 1, 2, \dots$.
This can be proved by induction. It is indeed true for $a_1 = \sqrt{2}$. Next, if we assume that $0 < a_n < 2$, then $0 < a_{n+1} = \sqrt{2 + a_n} < \sqrt{2 + 2} = \sqrt{4} = 2$.
2. The sequence is increasing. In fact: $a_{n+1}^2 = 2 + a_n > a_n + a_n = 2a_n > a_n^2$, hence $a_{n+1} > a_n$.

According to the *Monotonic Sequence Theorem*, every bounded monotonic (increasing or decreasing) sequence has a limit, hence a_n must have in fact a limit $L = \lim_{n \rightarrow \infty} a_n$.

Now that we know that the sequence has a limit L , by taking limits in the recursive relation $a_{n+1} = \sqrt{2 + a_n}$, we get $L = \sqrt{2 + L}$, hence $L^2 - L - 2 = 0$, so $L = 2$ or -1 . Since $a_n > 0$ then $L \geq 0$, hence $L = 2$. Consequently:

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}} = 2.$$

- 8.3.** We will prove that the answer is $(3 + \sqrt{5})/2$.

The value of the infinite continued fraction is the limit L of the sequence defined recursively $x_0 = 2207$, $x_{n+1} = 2207 - 1/x_n$, which exists because the sequence is decreasing (induction). Taking limits in both sides we get that $L = 2007 - 1/L$.

Since $x_n > 1$ for all n (also proved by induction), we have that $L \geq 1$. If we call the answer r we have $r^8 = L$, so $r^8 + 1/r^8 = 2207$. Then $(r^4 + 1/r^4)^2 = r^8 + 2 + 1/r^8 = 2 + 2207 = 2209$, hence $r^4 + 1/r^4 = \sqrt{2209} = 47$. Analogously, $(r^2 + 1/r^2)^2 = r^4 + 2 + 1/r^4 = 2 + 47 = 49$, so $r^2 + 1/r^2 = \sqrt{49} = 7$. And $(r + 1/r)^2 = r^2 + 2 + 1/r^2 = 2 + 7 = 9$, so $r + 1/r = \sqrt{9} = 3$. From here we get $r^2 - 3r + 1 = 0$, hence $r = (3 \pm \sqrt{5})/2$, but $r = L^{1/8} \geq 1$, so $r = (3 + \sqrt{5})/2$.

- 8.4.** That function coincides with $g(x) = 1/(1+x^2)$ at the points $x = 1/n$, and the derivatives of g at zero can be obtained from its Maclaurin series $g(x) = 1 - x^2 + x^4 - x^6 + \dots$, namely $g^{(2k)}(0) = (-1)^k k!$ and $g^{(2k+1)}(0) = 0$. In order to prove that the result applies to f too we have to study their difference $h(x) = f(x) - g(x)$. We have that $h(x)$ is infinitely differentiable. Also $h(1/n) = 0$ for $n = 1, 2, 3, \dots$, hence $h(0) = \lim_{n \rightarrow \infty} h(1/n) = 0$. By Rolle's theorem, $h'(x)$ has zeros between the zeros of $h(x)$, hence $h'(0)$ is the limit of a sequence of zeros, so $h'(0) = 0$. The same is true about all derivatives of h at zero. This implies that $f^{(k)}(0) = g^{(k)}(0)$ for every $k = 1, 2, 3, \dots$, hence $f^{(2k)}(0) = (-1)^k k!$ and $f^{(2k+1)}(0) = 0$.

- 8.5.** By looking at the graph of the function $y = 1/x$ we can see that

$$\int_n^{2n} \frac{1}{x} dx < \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n-1} < \int_{n-1}^{2n-1} \frac{1}{x} dx.$$

We have

$$\begin{aligned} \int_n^{2n} \frac{1}{x} dx &= \ln(2n) - \ln n = \ln 2, \\ \int_{n-1}^{2n-1} \frac{1}{x} dx &= \ln(2n-1) - \ln(n-1) = \ln \left\{ \frac{2n-1}{n-1} \right\} \xrightarrow{n \rightarrow \infty} \ln 2. \end{aligned}$$

Hence by the Squeeze Theorem, the desired limit is $\ln 2$.

- 8.6.** Let P be the limit. Then

$$\ln(P) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n} \ln \left(1 + \frac{k}{n} \right)$$

That sum is a Riemann sum for the following integral:

$$\int_0^1 \ln(1+x) dx = [(1+x)(\ln(1+x) - 1)]_0^1 = 2 \ln 2 - 1.$$

Hence $P = e^{2 \ln 2 - 1} = 4/e$.

- 8.7.** The series on the left is $xe^{-x^2/2}$. Since the terms of the second sum are non-negative, we can interchange the sum and integral:

$$\int_0^\infty xe^{-x^2/2} \sum_{n=0}^\infty \frac{x^{2n}}{2^{2n}(n!)^2} dx = \sum_{n=0}^\infty \int_0^\infty xe^{-x^2/2} \frac{x^{2n}}{2^{2n}(n!)^2} dx$$

The term for $n = 0$ is

$$\int_0^\infty x e^{-x^2/2} dx = \left[-e^{-x^2/2} \right]_0^\infty = 0 - (-1) = 1.$$

Next, for $n \geq 1$, integrating by parts:

$$\int_0^\infty x^{2n} \left(x e^{-x^2/2} \right) dx = \underbrace{\left[-x^{2n} e^{-x^2/2} \right]_0^\infty}_0 + 2n \int_0^\infty x^{2(n-1)} \left(x e^{-x^2/2} \right) dx.$$

Thus, by induction

$$\int_0^\infty x^{2n} \left(x e^{-x^2/2} \right) dx = 2 \cdot 4 \cdot 6 \cdots 2n.$$

Hence the integral is

$$\sum_{n=0}^\infty \frac{1}{2^n n!} = e^{1/2} = \sqrt{e}.$$

- 8.8.** The answer is affirmative, in fact *any* real number r is the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$. First assume

$$\sqrt[3]{n} \leq r + \sqrt[3]{m} < \sqrt[3]{n+1},$$

which can be accomplished by taking $n = \lfloor (r + \sqrt[3]{m})^3 \rfloor$. We have

$$\begin{aligned} 0 &\leq r - (\sqrt[3]{n} - \sqrt[3]{m}) < \sqrt[3]{n+1} - \sqrt[3]{n} \\ &= \frac{1}{\sqrt[3]{(n+1)^2} + \sqrt[3]{(n+1)n} + \sqrt[3]{n^2}}. \end{aligned}$$

Since the last expression tends to 0 as $n \rightarrow \infty$, we have that

$$r = \lim_{m \rightarrow \infty} \left\{ \sqrt[3]{\lfloor (r + \sqrt[3]{m})^3 \rfloor} - \sqrt[3]{m} \right\}.$$

- 8.9.** From $f(f(x)) = 1/f(x)$ we get that $f(y) = 1/y$ for all $y \in f(\mathbb{R})$. Hence $f(999) = 1/999$. Since f is continuous it takes all possible values between $1/999$ and 999 , in particular $500 \in f(\mathbb{R})$. Hence $f(500) = 1/500$.

- 8.10.** Consider the function $g : [0, 1998/1999] \rightarrow \mathbb{R}$, $g(x) = f(x) - f(x + 1/1999)$. Then g is continuous on $[0, 1998/1999]$, and verifies

$$\sum_{k=0}^{1998} g(k/1999) = f(1) - f(0) = 0.$$

Since the sum is zero it is impossible that all its terms are positive or all are negative, so either one is zero, or there are two consecutive terms with opposite signs. In the former case, $g(k/1999) = 0$ for some k , so $f(k/1999) = f((k+1)/1999)$ and we are done. Otherwise, if there are two consecutive terms $g(k/1999)$ and $g((k+1)/1999)$ with different signs, then for some $x \in [k/1999, (k+1)/1999]$ we have $g(x) = 0$, hence $f(x) = f(x + 1/1999)$, and we are also done.

- 9.1.** We divide the set into n -classes $\{1, 2\}, \{3, 4\}, \dots, \{2n-1, 2n\}$. By the pigeonhole principle, given $n+1$ elements, at least two of them will be in the same class, $\{2k-1, 2k\}$ ($1 \leq k \leq n$). But $2k-1$ and $2k$ are relatively prime because their difference is 1.
- 9.2.** For each odd number $\alpha = 2k-1$, $k = 1, \dots, n$, let C_α be the set of elements x in S such that $x = 2^i \alpha$ for some i . The sets $C_1, C_3, \dots, C_{2n-1}$ are a classification of S into n classes. By the pigeonhole principle, given $n+1$ elements of S , at least two of them will be in the same class. But any two elements of the same class C_α verify that one is a multiple of the other one.
- 9.3.** The given set can be divided into 18 subsets $\{1\}, \{4, 100\}, \{7, 97\}, \{10, 94\}, \dots, \{49, 55\}, \{52\}$. By the pigeonhole principle two of the numbers will be in the same set, and all 2-element subsets shown verify that the sum of their elements is 104.
- 9.4.** For $k = 1, 2, \dots, 8$, look at the digit used in place k for each of the 4 given elements. Since there are only 3 available digits, two of the elements will use the same digit in place k , so they coincide at that place. Hence at each place, there are at least two elements that coincide at that place. Pick any pair of such elements for each of the 8 places. Since there are 8 places we will have 8 pairs of elements, but there are only $\binom{4}{2} = 6$ two-element subsets in a 4-element set, so two of the pairs will be the same pair, and the elements of that pair will coincide in two different places.
- 9.5.** Let a_j the number of games played from the 1st through the j th day of the month. Then $a_1, a_2, \dots, a_{30}$ is an increasing sequence of distinct positive integers, with $1 \leq a_j \leq 45$. Likewise, $b_j = a_j + 14$, $j = 1, \dots, 30$ is also an increasing sequence of distinct positive integers with $15 \leq b_j \leq 59$. The 60 positive integers $a_1, \dots, a_{30}, b_1, \dots, b_{30}$ are all less than or equal to 59, so by the pigeonhole principle two of them must be equal. Since the a_j 's are all distinct integers, and so are the b_j 's, there must be indices i and j such that $a_i = b_j = a_j + 14$. Hence $a_i - a_j = 14$, i.e., exactly 14 games were played from day $j+1$ through day i .

10.1. We have

$$\frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)} = \frac{3^k}{3^k - 2^k} - \frac{3^{k+1}}{3^{k+1} - 2^{k+1}}.$$

So this is a telescopic sum:

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)} &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left\{ \frac{3^k}{3^k - 2^k} - \frac{3^{k+1}}{3^{k+1} - 2^{k+1}} \right\} \\ &= \lim_{n \rightarrow \infty} \left\{ 3 - \frac{3^{n+1}}{3^{n+1} - 2^{n+1}} \right\} \\ &= 3 - 1 = 2. \end{aligned}$$

10.2. This is a telescopic product:

$$\frac{n^3 - 1}{n^3 + 1} = \frac{(n-1)(n^2 + n + 1)}{(n+1)(n^2 - n + 1)} = \frac{(n-1)\{n(n+1) + 1\}}{(n+1)\{(n-1)n + 1\}},$$

hence

$$\begin{aligned} \prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1} &= \lim_{N \rightarrow \infty} \prod_{n=2}^N \frac{(n-1)\{n(n+1) + 1\}}{(n+1)\{(n-1)n + 1\}} \\ &= \lim_{N \rightarrow \infty} \frac{2\{N(N+1) + 1\}}{3N(N+1)} = \frac{2}{3}. \end{aligned}$$

11.1. Consider the function

$$f(x, y, z) = T(x, y, z) + T(y, z, x) + T(z, x, y) = 4x^2 + 4y^2 + 4z^2.$$

On the surface of the planet that function is constant and equal to $4 \cdot 20^2 = 1600$, and its average on the surface of the planet is 1600. Since the equation of a sphere with center in $(0, 0, 0)$ is invariant by rotation of coordinates, the three terms $T(x, y, z)$, $T(y, z, x)$, $T(z, x, y)$ have the same average value $\bar{T}$ on the surface of the planet, hence $1600 = 3\bar{T}$, and $\bar{T} = 1600/3$.

11.2. Writing $\alpha = \sqrt{2}$, the integrand $f(x) = 1/(1 + \tan^\alpha x)$ verifies the following symmetry:

$$\begin{aligned} f(x) + f\left(\frac{\pi}{2} - x\right) &= \frac{1}{1 + \tan^\alpha x} + \frac{1}{1 + \cot^\alpha x} \\ &= \frac{1}{1 + \tan^\alpha x} + \frac{\tan^\alpha x}{1 + \tan^\alpha x} \\ &= 1. \end{aligned}$$

On the other hand, making the substitution $u = \frac{\pi}{2} - x$:

$$\int_0^{\frac{\pi}{2}} f\left(\frac{\pi}{2} - x\right) dx = - \int_{\frac{\pi}{2}}^0 f(u) du = \int_0^{\frac{\pi}{2}} f(x) dx = I,$$

where I is the desired integral. So:

$$2I = \int_0^{\frac{\pi}{2}} \{f(x) + f\left(\frac{\pi}{2} - x\right)\} dx = \int_0^{\frac{\pi}{2}} 1 dx = \frac{\pi}{2}.$$

Hence $I = \frac{\pi}{4}$.

12.1. Let A be the set of positive integers no exceeding 1000 that are divisible by 7, and let B the set of positive integers not exceeding 1000 that are divisible by 11. Then $A \cup B$ is the set of positive integers not exceeding 1000 that are divisible by 7 or 11. The number of elements in A is $|A| = \left\lfloor \frac{1000}{7} \right\rfloor = 142$. The number of

elements in B is $|B| = \left\lfloor \frac{1000}{11} \right\rfloor = 90$. The set of positive integers not exceeding 1000 that are divisible by 7 and 11 is $A \cap B$, and the number of elements in there is $|A \cap B| = \left\lfloor \frac{1000}{7 \cdot 11} \right\rfloor = 12$. Finally, by the inclusion-exclusion principle:

$$|A \cup B| = |A| + |B| - |A \cap B| = 142 + 90 - 12 = 220.$$

- 12.2.** (This is equivalent to finding the number of onto functions from a n -element set to an k -element set.) If we remove the restriction "using all k flavors" then the first child can receive an ice-cream of any of the k available flavors, the same is true for the second child, and the third, etc. Hence the number of ways will be the product $k \cdot k \cdots k = k^n$.

Now we need to eliminate the distributions of ice-cream cones in which at least one of the flavors is unused. So let's call A_i = set of distributions of ice-creams in which at least the i th flavor is never used. We want to find the number of elements in the union of the A_i 's (and later subtract it from k^n). According to the Principle of Inclusion-Exclusion that number is the sum of the elements in each of the A_i 's, minus the sum of the elements of all possible intersections of two of the A_i 's, plus the sum of the elements in all possible intersections of three of those sets, and so on. We have:

$$|A_i| = (k-1)^n \text{ (} k-1 \text{ flavors distributed among } n \text{ children)}$$

$$|A_i \cap A_j| = (k-2)^n \text{ (} k-2 \text{ flavors among } n \text{ children)}$$

$$|\text{any triple intersection}| = (k-3)^n \text{ (} k-3 \text{ flavors among } n \text{ children)}$$

and so on. On the other hand there are k sets A_i , $\binom{k}{2}$ intersections of two sets, $\binom{k}{3}$ intersections of three sets, etc. Hence the number of distributions of flavors that miss some flavor is

$$\binom{k}{1}(k-1)^n - \binom{k}{2}(k-2)^n + \binom{k}{3}(k-3)^n - \cdots \pm \binom{k}{k}0^n,$$

and the number of distributions of flavors that do not miss any flavor is k^n minus the above sum, i.e.:

$$k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - \binom{k}{3}(k-3)^n + \cdots \mp \binom{k}{k}0^n = \sum_{i=0}^k (-1)^i \binom{k}{i} (k-i)^n.$$

- 12.3.** Let's denote P_i the set of permutations fixing element a_i . The set of non-derangements are the elements of the union $P_1 \cup P_2 \cup \cdots \cup P_n$, and its number can be found using

the inclusion-exclusion principle:

$$|P_1 \cup P_2 \cup \cdots \cup P_n| = \sum_i |P_i| - \sum_{i \neq j} |P_i \cap P_j| + \sum_{i \neq j \neq k \neq i} |P_i \cap P_j \cap P_k| - \cdots$$

Each term of that expression is the number of permutations fixing a certain number of elements. The number of permutations that fix m given elements is $(n - m)!$, and since there are $\binom{n}{m}$ ways of picking those m elements, the corresponding sum is $\binom{n}{m}(n - m)! = \frac{n!}{m!}$. Adding and subtracting from the total number of permutations $n!$, we get

$$D_n = n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + (-1)^n \frac{n!}{n!} = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!} \right).$$

13.1. Let
$$I = \frac{10^{20000} - 3^{200}}{10^{100} + 3} = \frac{(10^{100})^{200} - 3^{200}}{10^{100} + 3} = (10^{100})^{199} - (10^{100})^{198} \cdot 3 + \cdots + 10^{100} \cdot 3^{198} - 3^{199}$$

so I is an integer. On the other hand since $\frac{3^{200}}{10^{100}+3} < 1$ we have that $\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor = I$. Finally the rightmost digit of I can be found as the 1-digit number congruent to $-3^{199} \pmod{10}$. The sequence $3^n \pmod{10}$ has period 4 and $199 = 3 + 4 \cdot 49$, hence $-3^{199} \pmod{10} = -3^3 \pmod{10} = -27 \pmod{10} = 3$. Hence the units digit of I is 3.

13.2. Let α be any (say the smallest) acute angle of a right triangle with sides 3, 4 and 5 (or any other Pythagorean triple). Next, place an infinite sequence of points on the unit circle at coordinates $(\cos(2n\alpha), \sin(2n\alpha))$, $n = 0, 1, 2, \dots$ (The sequence contains in fact infinitely many points because α cannot be a rational multiple of π .) The distance from $(\cos(2n\alpha), \sin(2n\alpha))$ to $(\cos(2m\alpha), \sin(2m\alpha))$ is $2\sin(|n - m|\alpha)$, so all we need to prove is that $\sin(k\alpha)$ is rational for any k . This can be done by induction using that $\sin \alpha$ and $\cos \alpha$ are rational, and if $\sin u$, $\cos u$, $\sin v$ and $\cos v$ are all rational so are $\sin(u + v) = \sin u \cos v + \cos u \sin v$ and $\cos(u + v) = \cos u \cos v - \sin u \sin v$.

13.3. We can prove the first part by way of contradiction. Assume that we have colored the points of the plane with three colors such that any two points at distance 1 have different colors. Consider any two points A and B at distance $\sqrt{3}$ (see figure 3). The circles of radius 1 and center A and B meet at two points P and Q , forming equilateral triangles APQ and BPQ . Since the vertices of each triangle must have different colors that forces A and B to have the same color. So any two points at distance $\sqrt{3}$ have the same color. Next consider a triangle DCE with $CD = CE = \sqrt{3}$ and $DE = 1$. The points D and E must have the same color as C , but since they are at distance 1 they should have different colors, so we get a contradiction.

For the second part, if we replace “three” by “nine” then we can color the plane with nine different colors so that any two points at distance 1 have different colors: we can arrange them periodically in a grid of squares of size $2/3 \times 2/3$ as shown in figure 4. If two points P and Q have the same color then either they belong to

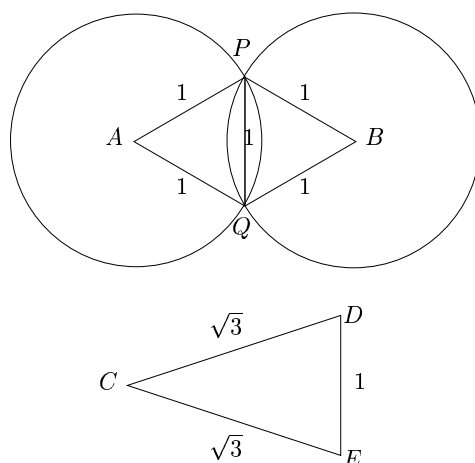


FIGURE 3

the same square and $PQ < (2/3)\sqrt{2} < 1$, or they belong to different squares and $PQ \geq 4/3 > 1$.

	A	B	C	A	B	C	
	D	E	F	D	E	F	
	G	H	I	G	H	I	
	A	B	C	A	B	C	
	D	E	F	D	E	F	

FIGURE 4

- 13.4.** Since the values are positive integers, one of them, say n , will be the smallest one. Look at any square with that value n . Since the values of its four neighbors must be at least n and their average is n , all four will have value n . By the same reasoning

the neighbors of these must be n too, and so on, so all the squares must have the same value n .

- 13.5.** Choose $\alpha \in \mathbb{R}$ so that α^3 is irrational, for example $\alpha = \sqrt[3]{2}$. Use punches at $A = (-\alpha, 0)$, $B = (0, 0)$, and $C = (\alpha, 0)$. If $P = (x, y)$ then

$$AP^2 - 2BP^2 + CP^2 = (x + \alpha)^2 + y^2 - 2(x^2 + y^2) + (x - \alpha)^2 + y^2 = 2\alpha^2$$

is irrational, so AP , BP , CP cannot all be rational.

- 13.6.** We have

$$f(n+2) - f(n+1) = (n+2)! = (n+2)(n+1)! = (n+2)(f(n+1) - f(n)),$$

hence

$$\begin{aligned} f(n+2) &= (n+2)(f(n+1) - f(n)) + f(n+1) \\ &= (n+3)f(n+1) - (n+2)f(n), \end{aligned}$$

and we can take $P(x) = x + 3$, $Q(x) = -x - 2$.

- 13.7.** A conspiratorial subset of $S = \{1, 2, \dots, 16\}$ has at most two elements from $T = \{1, 2, 3, 5, 7, 11, 13\}$, so it has at most $2 + 16 - 7 = 11$ numbers. On the other hand all elements of $S \setminus T = \{4, 6, 8, 9, 10, 12, 14, 15, 16\}$ are multiple of either 2 or 3, so adding 2 and 3 we obtain the following 11-element conspiratorial subset:

$$\{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16\}.$$

Hence the answer is 11.

- 13.8.** The statement is true. Let ϕ any bijection on F with no fixed points ($\phi(x) \neq x$ for every x), and set $x * y = \phi(x)$. Then

- (i) $x * z = y * z$ means $\phi(x) = \phi(y)$, and this implies $x = y$ because ϕ is a bijection.
- (ii) We have $x * (y * z) = \phi(x)$ and $(x * y) * z = \phi(\phi(x))$, which cannot be equal because that would imply that $\phi(x)$ is a fixed point of ϕ .