

The power mean inequality

Let $x_1, \dots, x_n \geq 0$. For $r \neq 0$ (assume $r > 0$ if some x_i are zero), the r -th power mean P_r of $x_1, \dots, x_n$ is defined to be the r -th root of the average of the r -th powers of $x_1, \dots, x_n$:

$$P_r := \left(\frac{x_1^r + \dots + x_n^r}{n} \right)^{\frac{1}{r}}.$$

This formula is not defined if $r = 0$, but there is a natural way to define P_0 too: it is simply defined to be the *geometric mean*: $P_0 := \sqrt[n]{x_1 x_2 \dots x_n}$.

We can also define $P_\infty := \max \{x_1, \dots, x_n\}$

since when r is very large, P_r is a good approximation to the largest of $x_1, \dots, x_n$. For a similar reason we can use the notation $P_{-\infty} := \min \{x_1, \dots, x_n\}$.

Here are some examples:

$$P_1 = \frac{x_1 + \dots + x_n}{n} \quad \text{is the } \textit{arithmetic mean},$$

$$P_2 = \sqrt{\frac{x_1^2 + \dots + x_n^2}{n}} \quad \text{is called the } \textit{quadratic mean}.$$

For $x_1, \dots, x_n > 0$,

$$P_{-1} = \frac{n}{\frac{1}{x_1} + \dots + \frac{1}{x_n}} \quad \text{is called the } \textit{harmonic mean}.$$

Power mean inequality:

Let $x_1, \dots, x_n \geq 0$. Suppose $r > s$ (and $s \geq 0$ if any of the x_i are zero). Then $P_r \geq P_s$, with equality if and only if $x_1 = x_2 = \dots = x_n$.

The power mean inequality holds even if $r = \infty$ or $s = -\infty$, provided that we use the definitions of P_∞ and $P_{-\infty}$ above, and the convention that $\infty > r > -\infty$ for all numbers r .

Here are some special cases of the power mean inequality:

- $P_1 \geq P_0$ (the AM-GM inequality).
- $P_0 \geq P_{-1}$ (the GM-HM inequality -- HM is for "harmonic mean").
- $P_1 \geq P_{-1}$ (the AM-HM inequality).