

The Pigeon Hole Principle

If you have fewer pigeon holes than pigeons and you put every pigeon in a pigeon hole, then there must result at least one pigeon hole with more than one pigeon.

(Extended) Pigeon-Hole Principle.

If there are k pigeon-holes and more than mk pigeons then at least one pigeon-hole has at least $m + 1$ pigeons.

Examples

1. Suppose that in my dresser drawer I have socks of three colours ... loose. How do I ensure that I get a matching pair of socks in the most economical way?

Solution. I take 4 socks from the drawer...since then, by the PHP, I must have at least one pair. The idea is that the colours (three of them) are the pigeon-holes and the socks are the pigeons.

2. Among any N positive integers, there exists 2 whose difference is divisible by $N-1$.

Solution. Let $a_1, a_2, \dots, a_N$ be the numbers. For each a_i , let r_i be the remainder that results from dividing a_i by $N - 1$. (So $r_i = a_i \bmod (N-1)$ and r_i can take on only the values $0, 1, \dots, N-2$.) There are $N-1$ possible values for each r_i , but there are N r_i 's. Thus, by the pigeon hole principle, there must be two of the r_i 's that are the same, $r_j = r_k$ for some pair j and k . But then, the corresponding a_i 's have the same remainder when divided by $N-1$, and so their difference $a_j - a_k$ is evenly divisible by $N-1$.

3. If we take an arbitrary subset A of $n + 1$ integers from the set $[2n] = \{1, \dots, 2n\}$ it will contain a pair of co-prime integers.

Solution. Define the holes as sets $\{1, 2\}, \{3, 4\}, \dots, \{2n-1, 2n\}$. Thus n holes are defined.

If we place the $n + 1$ integers of A into their corresponding holes – by the pigeon-hole principle – there will be a hole, which will contain two numbers. This means, that A has to contain two consecutive integers, say, x and $x + 1$. But two such numbers are always co-prime.

If some integer $y > 1$ divides x , i.e., $x = ky$, then $x + 1 = ky + 1$ and this is not divisible by y .

4. Let $P_1, P_2, \dots, P_n$ be n points in the unit square $[0, 1]^2$. Show that there exist $i, j, k \in \{1, \dots, n\}$

such that the triangle $P_i P_j P_k$ has area $\leq \frac{1}{2 \left(\left\lfloor \sqrt{\frac{n-1}{2}} \right\rfloor \right)^2} \sim \frac{1}{n}$ for large n .

Solution. Let $m = \left\lfloor \sqrt{\frac{(n-1)}{2}} \right\rfloor$ and divide the square up into $m^2 < \frac{n}{2}$ subsquares. By the PHP,

there must be a square containing ≥ 3 points. Let 3 of these points be $P_i P_j P_k$.

The area of the corresponding triangle is at most one half of the area of an individual square.