

Pascal Triangle and Binomial Formula (Expansion)

Pascal Triangle:	1							Row 0	
	1			1				1-st row	
		1	2	1				2-nd row	
		1	3	3	1			3-rd row	
	1	1	4	6	4	1			
	1	5	10	10	5	1			
	...							.	
							.		
							.		
	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	...	$\binom{n}{k}$	...	$\binom{n}{n-1}$	$\binom{n}{n}$	n-th row

Some properties of Pascal Triangle:

- The **Newton's bracket** $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$ represents the number of combinations of choosing k -element subsets from an n -element set.
- The entries in n -th row of Pascal Triangle are used for coefficients in **Binomial Formula**:

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

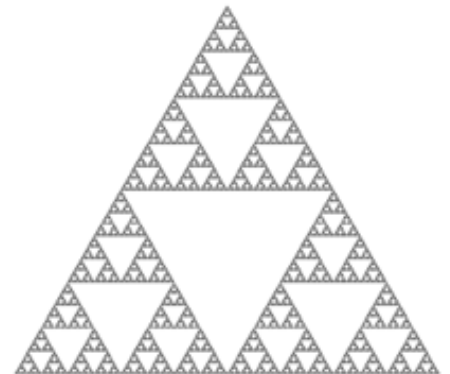
- The sum of all the entries in n -th row represents the number of all subsets of an n -element set:

$$\sum_{k=0}^n \binom{n}{k} = 2^n$$

- Pascal Triangle is build by adding the consecutive entries in the previous row to make the entry in the next row, so

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

- The pattern obtained by colouring only the odd numbers in Pascal's triangle closely resembles the **Sierpinski Triangle**:



- The sum of the squares of the entries of n -th row equals the middle

entry in the $(2n)$ -th row: $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

- The sum of the entries in the first $(n-1)$ rows (together with the row 0) is the n -th **Mersenne number**:

$$\sum_{m=0}^{n-1} \sum_{k=0}^m \binom{m}{k} = 2^n - 1 \quad (\text{The } n\text{-th Mersenne number is prime if } n \text{ is prime.})$$

Problems involving Binomial Expansion:

1. Show that $\sum_{k=0}^n \binom{n}{k} = 2^n$ and $\sum_{k=0}^n (-1)^k \binom{n}{k} = 0$.
2. Show that $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$ (Hint: consider $(1+x)^n (1+x)^n = (1+x)^{2n}$)
3. Find $\sum_{k=0}^n 2^k \binom{n}{k}$ (Hint: consider binomial $(1+x)^n$)
4. Which is larger, 1999^{1999} or 2000^{1998} ?
5. Show that 7 divides $6^{2n} - 1$ for every natural n .
6. Show that for natural $n > 1$ and $1 \leq k \leq n-1$, the binomial coefficient $\binom{n}{k}$ is divisible by n if and only if n is prime.
7. Prove that $(a+b)^p - a^p - b^p$ is divisible by p if p is prime and a, b are any integers.
8. Prove that for a prime $p > 2$, $2^{p-1} - 1$ is divisible by p .
9. Let a, b be integers not divisible by a prime number p . Prove that if $a^p = b^p \pmod{p}$, then $a^p = b^p \pmod{p^2}$. (Hint: Use **Fermat's Little Theorem** stating that if p prime then $a^p = a \pmod{p}$ for any integer a .)
10. (2004B2) Let m, n be positive integers. Show that $\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!n!}{m^m n^n}$.