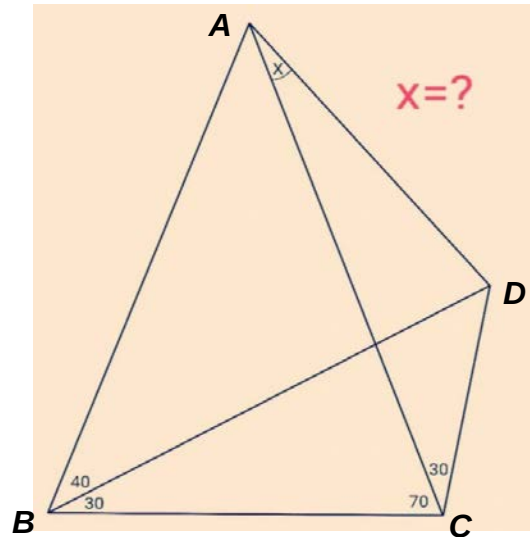


Math Match 10 Problems:

- Find the measure of angle x .



Solution:

Let E be the circumcenter of $\triangle BCD$. Notice that $\angle CED = 2 \cdot \angle CBD = 60$ so $\triangle ECD$ is equilateral. Thus $\angle ECA = 30$ and $\triangle EAC \cong \triangle DAC$, so $\angle EAC = \angle DAC = x$. Also $\triangle AEB \cong \triangle AEC$ (by SSS), so $\angle EAB = \angle EAC = x$. Since $\angle BAC = 40 = 2x$, then $x = 20$.

- Find all positive integers n such that $2n + 1$ divides $n^3 - n + 3$.

Solution:

Consider polynomial $8(n^3 - n + 3)$. Notice that $2n + 1$ divides $n^3 - n + 3$ iff $2n + 1$ divides $8(n^3 - n + 3)$. Since $8(n^3 - n + 3) = (2n + 1)(4n^2 - 2n - 3) + 27$ then $2n + 1$ must divide 27. This happens for $n = 1, 4$, or 13 .

- Find the smallest natural number n such that there exists a choice of the signs that makes the following equality true $\pm 1 \pm 2 \pm 3 \pm \dots \pm n = 2025$

Solution:

Notice that n must be bigger than 63 as $\sum_{k=1}^{63} k = \frac{63 \cdot 64}{2} = 2016 < 2025$.

$n = 64$ wouldn't work as the corresponding sum, independent of the choice of signs, would be even (the sum with all positive signs is even and we subtract twice some amount to accommodate for the negative signs).

For $n=65$, we have $2025 = \sum_{k=1}^{65} k - 2(60)$, so $n = 65$ is the smallest solution.

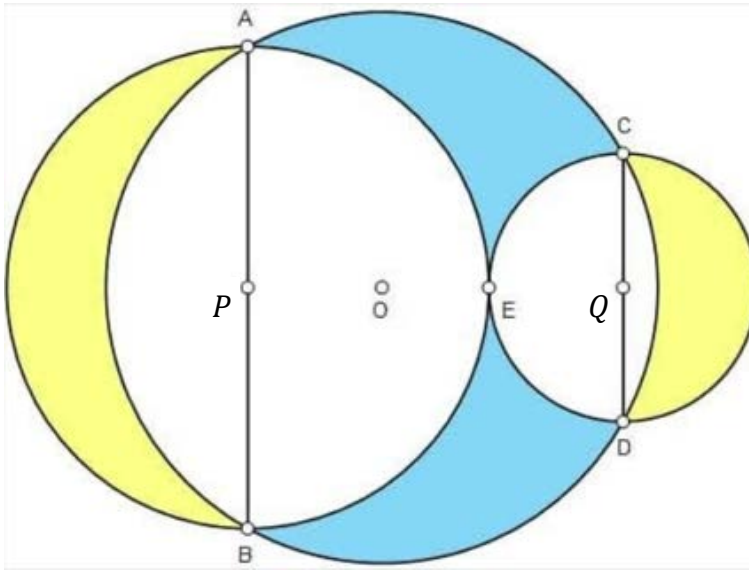
- Let p be a prime number greater than 5. Prove that $p - 4$ is not a 4th power of any integer.

Solution:

Towards a contradiction, assume that $p - 4 = n^4$ for some integer $n > 1$.

Then $p = n^4 + 4 = (n^2 + 2)^2 - 4n^2 = (n^2 + 2 - 2n)(n^2 + 2 + 2n)$ and both factors are greater than 1. This contradicts the fact that p is prime.

5.



Given: Circles P and Q with diameters AB and CD , tangent at E . Circle O passes through points A, B, C, D .

Prove: Yellow shaded area = Blue shaded area

Solution:

Let $OA = R, PE = r_1, QE = r_2$.

First, we show that $PO = r_2$.

Using the right-angle triangle OPA , we have $R^2 = r_1^2 + (r_1 - OE)^2$

Using the right-angle triangle OQC , we have $R^2 = r_2^2 + (r_2 + OE)^2$

This gives us $2r_1^2 - 2r_1OE + OE^2 = 2r_2^2 + 2r_2OE + OE^2$, which simplifies to

$r_1^2 - r_2^2 = (r_2 + r_1)OE$ and finally $r_1 - r_2 = OE$. Thus $PO = r_1 - OE = r_2$

Now, substituting this to our first Pythagorean equation, we have $R^2 = r_1^2 + r_2^2$.

This means that the area of circle O is equal to the sum of areas of circles P and Q . Since the blue area is obtained by subtracting the white area from the area of circle O and the yellow area is obtained by subtracting the same white area from the sum of areas of circles P and Q , then the blue and yellow areas are equal.

6. Let $(x_n)_{n \geq 1}$ be a sequence of real numbers such that $\lim_{n \rightarrow \infty} \frac{x_1^2 + x_2^2 + \dots + x_n^2}{n} = 0$.

a) Show that $\lim_{n \rightarrow \infty} \frac{x_1 + x_2 + \dots + x_n}{n} = 0$.

b) Is the converse true? (Note: Converse to the statement $p \Rightarrow q$ is $q \Rightarrow p$.)

Solution:

a) By arithmetic-square mean inequality, we have $\frac{x_1 + x_2 + \dots + x_n}{n} \leq \sqrt{\frac{x_1^2 + x_2^2 + \dots + x_n^2}{n}}$ which results in

$$0 \leq \left(\frac{x_1 + x_2 + \dots + x_n}{n} \right)^2 \leq \frac{x_1^2 + x_2^2 + \dots + x_n^2}{n}$$

(Alternatively, we can use Cauchy-Swartz inequality ($|\langle u, v \rangle| \leq \|u\| \|v\|$) for vectors $u = \langle x_1, x_2, \dots, x_n \rangle$ and $v = \langle 1, 1, \dots, 1 \rangle$ to obtain the above inequality.)

Since $\lim_{n \rightarrow \infty} \frac{x_1^2 + x_2^2 + \dots + x_n^2}{n} = 0$, then by Squeeze Theorem $\lim_{n \rightarrow \infty} \left(\frac{x_1 + x_2 + \dots + x_n}{n} \right)^2 = 0$ which

implies that $\lim_{n \rightarrow \infty} \frac{x_1 + x_2 + \dots + x_n}{n} = 0$

b) The converse is not true. Counterexample: Let $x_n = (-1)^n$. Then

$$\frac{x_1 + x_2 + \dots + x_n}{n} = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{1}{n} & \text{if } n \text{ is odd} \end{cases} \quad \text{Therefore } \lim_{n \rightarrow \infty} \frac{x_1 + x_2 + \dots + x_n}{n} = 0 \text{ but } \lim_{n \rightarrow \infty} \frac{x_1^2 + x_2^2 + \dots + x_n^2}{n} = 1.$$

7. Prove that there are no differentiable functions $f: \mathbb{R} \rightarrow \mathbb{R}$, such that both $f(x)$ and $f'(x)$ are integers only if x is an integer.

Solution:

Towards the contradiction, assume that such function exists. Let n be an integer. By Mean Value Theorem, there is a $c \in (n, n+1)$, such that $f'(c) = f(n+1) - f(n)$. Since $f'(c)$ is an integer and c isn't, we have a contradiction.

8. Let a and b be two positive integers such that both products ab and $(a+1)(b+1)$ are perfect squares. Show that there is an integer $n > 1$, such that $(a+n)(b+n)$ is also a perfect square.

Solution:

Notice that if $a = b = 1$ then $(a+n)(b+n) = (n+1)^2$. Otherwise, take $n = ab$. Then we have $(a+n)(b+n) = (a+ab)(b+ab) = ab \cdot (a+1)(b+1)$ which is a perfect square as the product of two perfect squares.