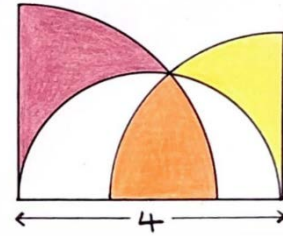


Problems for the 9th Annual Math Match 2024

- 1) Two quarter circles and a semicircle. What's the total shaded area?



- 2) Are there more positive integers under a hundred for which the nearest square is odd or for which it is even?
- 3) The equations $x^2 + ax + 1 = 0$ and $x^2 + bx + c = 0$ have a common real root, and the equations $x^2 + x + a = 0$ and $x^2 + cx + b = 0$ have a common real root. Find $a + b + c$.
- 4) Let a , b , and c be positive real numbers. Prove that

$$a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$$

- 5) Consider an equilateral triangle of side length n , divided into unit triangles, as shown. Let $f(n)$ be the number of paths from the triangle in the top row to the middle triangle in the bottom row, such that adjacent triangles in our path share a common edge and the path never travels up (from a lower row to a higher row) or revisits a triangle. An example of such path is illustrated below for $n = 5$. Determine the value of $f(2024)$.



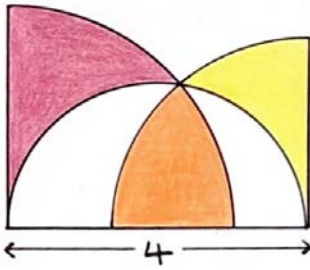
- 6) Let S be a set of $n \geq 3$ points in the interior of a circle.
- a) Show that there are three distinct points $a, b, c \in S$ and three distinct points A, B, C on the circle such that a is the closest point to A , b is the closest point to B , and c is the closest point to C out of all the points in S .
- b) Show that for no value of n can four such points in S (and corresponding points on the circle) be guaranteed.

Note: For the purpose of this Match it is OK to show solution to either part a) or b), not necessary both.

- 7) Can we place 65 balls of diameter 1 in a cubic box of the side equal to 4?
- 8) A convex quadrilateral $ABCD$ is inscribed in a circle and contains the circumcentre O in its interior. Suppose $\angle BAO = \angle DAC$. Prove that the diagonals of the quadrilateral are perpendicular.

Solutions:

1.



$$A + B - C$$

$$\begin{aligned} &= \frac{1}{4}\pi a^2 + \frac{1}{4}\pi b^2 - \frac{1}{2}\pi 2^2 \\ &= \frac{1}{4}\pi(a^2 + b^2) - \frac{1}{2}\pi 2^2 \\ &= \frac{1}{4}\pi 4^2 - \frac{1}{2}\pi 2^2 \\ &= 4\pi - 2\pi \\ &= 2\pi \end{aligned}$$

2. **Ans: Odd**

There are $2n$ integers for which the closest integer is n^2 , namely $n^2 - n + 1, n^2 - n + 2, \dots, n^2 + n$. So there are $2(1 + 3 + 5 + 7 + 9) = 50$ integers under a hundred for which the nearest square is odd. Hence there are 49 integers for which the nearest square is even.

3. **Ans: -3**

The common root of $x^2 + ax + 1 = 0$ and $x^2 + bx + c = 0$ must also satisfy $(a - b)x + (1 - c) = 0$, so it must be $\frac{c-1}{a-b}$. Note that the other root of $x^2 + ax + 1 = 0$ must be $\frac{a-b}{c-1}$, since the product of the roots is 1. Similarly, the common root of $x^2 + x + a = 0$ and $x^2 + cx + b = 0$ must satisfy $(c - 1)x + (b - a) = 0$, so it is $x = \frac{a-b}{c-1}$. Hence $x^2 + x + a = 0$ and $x^2 + ax + 1 = 0$ have a common root. Hence it satisfies $(a - 1)x + (1 - a) = 0$. Now we cannot have $a = 1$, for then $x^2 + ax + 1 = 0$ has no real roots. Hence the common root must be 1. Hence both roots of $x^2 + ax + 1 = 0$ are 1 and so $a = -2$.

So $x^2 + bx + c = 0$ has one root 1. Hence $b + c = -1$. Therefore, $a + b + c = -3$.

4. **Solution 1:** Equivalently, we prove that $a^{3a}b^{3b}c^{3c} \geq (abc)^{a+b+c}$. Due to symmetry, WLOG we can assume that $a \geq b \geq c$. Then, we have $a - b \geq 0, b - c \geq 0, a - c \geq 0$ and $\frac{a}{b} \geq 1, \frac{b}{c} \geq 1, \frac{a}{c} \geq 1$.

Since, we have $\frac{a^{3a}b^{3b}c^{3c}}{(abc)^{a+b+c}} = \frac{a^{2a}b^{2b}c^{2c}}{a^b a^c b^a b^c c^a c^b} = \frac{a^a b^b \cdot b^b c^c \cdot a^a c^c}{b^a a^b \cdot c^b b^c \cdot c^a a^c} = \left(\frac{a}{b}\right)^{a-b} \left(\frac{b}{c}\right)^{b-c} \left(\frac{a}{c}\right)^{a-c} \geq 1$

Solution 2: Assign the weights a, b, c to the numbers a, b, c , respectively. Then, by the weighted geometric - weighted harmonic mean inequality followed by the arithmetic - geometric mean inequality, we have

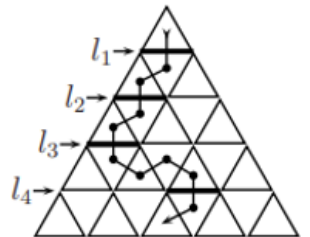
$$\sqrt[a+b+c]{a^a b^b c^c} \geq \frac{a + b + c}{\frac{a}{a} + \frac{b}{b} + \frac{c}{c}} = \frac{a + b + c}{3} \geq \sqrt[3]{abc},$$

So, $a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$.

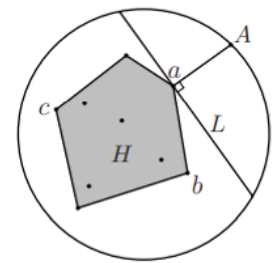
5. **Ans: $f(2024) = 2023!$**

We can show that $f(n) = (n - 1)!$. Label the horizontal line segments $l_1, l_2, \dots$ as in the diagram. Since the path travels from the top to the bottom row and never travels up, it must cross each of $l_1, l_2, \dots, l_{n-1}$ exactly once. Notice that the diagonal lines in the triangle divide each line segment l_k into k unit segments and the path must cross exactly one of these k segments.

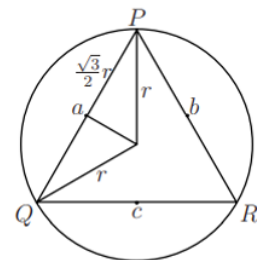
The path is completely determined by the crossings of the $n - 1$ line segments $l_1, l_2, \dots, l_{n-1}$. Since we have 1, 2, 3, ... $n - 1$ possible crossings of the corresponding line segments $l_1, l_2, \dots, l_{n-1}$, then the number of all possible paths is $1 \cdot 2 \cdot 3 \cdot \dots \cdot (n - 1) = n!$.



6. **Solution 1a:** Let H be the smallest convex set of points in the plane, containing S . Take 3 points $a, b, c \in S$ which lie on the boundary of H . Construct a chord L such that all points of S lie on one side of L . Let A be the point where the perpendicular to L meets the circle, on the opposite side (the side which does not contain S). By construction, A is closer to a than to any other point on the chord L or on the other side of L . Similarly, we can construct point B and C , which completes the proof.



Solution 1b: Let PQR be an equilateral triangle inscribed in the circle and a, b, c be midpoints of the 3 sides of PQR . If r is the radius of the circle, then every point on the circle is within $\frac{\sqrt{3}}{2}r < .9r$ of one of the points a, b , or c . If S consists of the 3 points a, b, c and a cluster of points within $0.1r$ from the centre of the circle, then we cannot select 4 points satisfying the conditions of the problem.



Solution 2a: If all the points of S lie on a line L , then choose any 3 of them to be a, b, c . Let A be a point on the circle where the perpendicular to L at a meets the circle. By construction, out of all points of S , a is the closest point to A . We construct points B and C similarly.

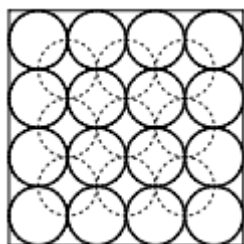
If not all points of S are collinear, choose a, b, c from S so that the area of a triangle abc is the greatest. Construct the altitude from the side bc to the point a and extend this line until it meets the circle at A . We claim that A is closer to a than to any other point in S .

Towards a contradiction, suppose a point $x \in S$ is closer to A than a . Then the distance of x from the line bc must be greater than the distance of a from bc . Hence the area of the triangle xbc would be greater than the area of the triangle abc , contradicting the original choice of these 3 points.

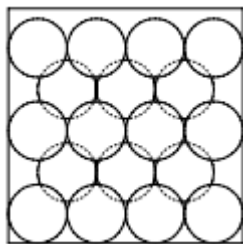
This proves that A is closer to a than to any other point of S . We construct points B and C analogously.

7. **Solution: Yes. We can place even 66 balls.**

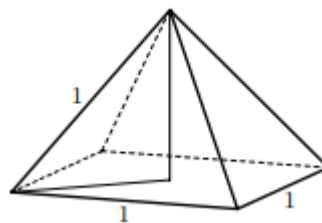
Place the first layer of 16 balls. Then the next layer of 9 balls. Then alternatively 16, 9, and 16. So, the total is $3 \cdot 16 + 2 \cdot 9 = 66$. Now we check the height of the five layers.



top view



side view



height between centers of circles in consecutive layers

A ball in the second layer is tangent to 4 balls in the first layer. Consider square pyramid with the vertices at the centres of the 5 balls. Using the Pythagorean Theorem, we can calculate the height of this pyramid to be $\frac{\sqrt{2}}{2}$. So, the height of the 5 layers is $\frac{1}{2} + 4 \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} = 1 + 2\sqrt{2} < 4$.

8. **Proof:** Observe that $\angle DAC = \angle BAO = \frac{\pi - \angle AOB}{2} = \frac{\pi}{2} - \angle ADB$.

So, $\angle DAC + \angle ADB = \frac{\pi}{2}$.

Therefore, $\angle DPA = \pi - (\angle DAP + \angle ADP) = \frac{\pi}{2}$.

